

**SOLUTIONS IV: ELECTRICITY MARKETS**  
 Will be worked on in the exercise session on Friday, 14 June 2019.

**SOLUTION IV.1 (SHADOW PRICES OF LIMITS ON CONSUMPTION).**

Suppose that the utility for the electricity consumption of an industrial company is given by

$$U(d) = 70d - 3d^2 [\text{€/MWh}] \quad , \quad d_{\min} = 2 \leq d \leq d_{\max} = 10,$$

where  $d$  is the demand in MW and  $d_{\min}, d_{\max}$  are the minimum and maximum demand.

Assume that the company is maximising its net surplus for a given electricity price  $\pi$ , i.e. it maximises  $\max_d [U(d) - \pi d]$ .

- (a) If the price is  $\pi = 5 \text{ €/MWh}$ , what is the optimal demand  $d^*$ ? What is the value of the KKT multiplier  $\mu_{\max}$  for the constraint  $d \leq d_{\max} = 10$  at this optimal solution? What is the value of  $\mu_{\min}$  for  $d \geq d_{\min} = 2$ ?

We convert the exercise to an optimisation problem with objective

$$\max_d U(d) - \pi d \tag{1}$$

with constraints

$$d \leq d_{\max} \quad \leftrightarrow \quad \mu_{\max} \tag{2}$$

$$-d \leq -d_{\min} \quad \leftrightarrow \quad \mu_{\min} \tag{3}$$

From stationarity we get:

$$0 = \frac{\partial}{\partial d} (U(d) - \pi d) - \mu_{\max} \frac{\partial}{\partial d} (d - d_{\max}) - \mu_{\min} \frac{\partial}{\partial d} (-d + d_{\min}) \tag{4}$$

$$= U'(d) - \pi - \mu_{\max} + \mu_{\min} \tag{5}$$

Note that it does not matter whether you pull in the constant of the right-hand side of the respective constraint.

The marginal utility curve is  $U'(d) = 70 - 6d \text{ [€/MWh]}$ . At  $\pi = 5$  and if the demand were unconstrained, the demand would be determined by  $5 = 70 - 6d$ , i.e.  $d = 65/6 = 10.8333$ , which is above the consumption limit  $d_{\max} = 10$ . Therefore the optimal demand is  $d^* = 10$ , the upper limit is binding such that  $\mu_{\max} \geq 0$  and the lower limit is non-binding such that  $\mu_{\min} = 0$ .

To determine the value of  $\mu_{\max}$  we use (5) to get  $\mu_{\max} = U'(d^*) - \pi + \mu_{\min} = U'(10) - 5 + 0 = 5$ .

(b) Suppose now the electricity price is  $\pi = 60$  €/MWh. What are the optimal demand  $d^*$ ,  $\mu_{max}$  and  $\mu_{min}$  now?

At  $\pi = 60$ , the demand would be determined by  $60 = 70 - 6d$ , i.e.  $d = 10/6 = 1.667$ , which is below the consumption limit  $d_{min} = 2$ . Therefore the optimal demand is  $d^* = 2$ , the upper limit is non-binding such that  $\mu_{max} = 0$  and the lower limit is binding such that  $\mu_{min} \geq 0$ .

To determine the value of  $\mu_{min}$  we use (5) to get  $\mu_{min} = \pi - U'(d^*) + \mu_{max} = 60 - U'(2) + 0 = 2$ .

#### SOLUTION IV.2 (ECONOMIC DISPATCH IN A SINGLE BIDDING ZONE).

Consider an electricity market with two generator types, one with the cost function  $C_1(g_1) = c_1 g_1$  with variable cost  $c_1 = 20$  €/MWh, capacity  $G_1 = 300$  MW and a dispatch rate of  $g_1$  [MW], and another with the cost function  $C_2(g_2) = c_2 g_2$  with variable cost  $c_2 = 50$  €/MWh, capacity  $G_2 = 400$  MW and a dispatch rate of  $g_2$  [MW]. The demand has utility function  $U(d) = 8000d - 5d^2$  [€/h] for a consumption rate of  $d$  [MW].

(a) What are the objective function and constraints required for an optimisation problem to maximise short-run social welfare in this market?

The optimisation problem has the objective function:

$$\max_{d, g_1, g_2} [U(d) - C_1(g_1) - C_2(g_2)] = \max_{d, g_1, g_2} [8000d - 5d^2 - c_1 g_1 - c_2 g_2]$$

with constraints:

$$\begin{aligned} d - g_1 - g_2 &= 0 \leftrightarrow \lambda \\ g_1 &\leq G_1 \leftrightarrow \bar{\mu}_1 \\ g_2 &\leq G_2 \leftrightarrow \bar{\mu}_2 \\ -g_1 &\leq 0 \leftrightarrow \underline{\mu}_1 \\ -g_2 &\leq 0 \leftrightarrow \underline{\mu}_2 \end{aligned}$$

(b) Write down the Karush-Kuhn-Tucker (KKT) conditions for this problem.

Stationarity gives for  $d$ :

$$\frac{\partial U}{\partial d} - \lambda = 8000 - 10d - \lambda = 0$$

Stationarity for  $g_1$  gives:

$$-\frac{\partial C_1}{\partial g_1} + \lambda - \mu_1 = -c_1 + \lambda - \bar{\mu}_1 + \underline{\mu}_1 = 0$$

Stationarity for  $g_2$  gives:

$$-\frac{\partial C_2}{\partial g_2} + \lambda - \mu_2 = -c_2 + \lambda - \bar{\mu}_2 + \underline{\mu}_2 = 0$$

Primal feasibility is just the generator limits above in (a). Dual feasibility is  $\bar{\mu}_i, \underline{\mu}_i \geq 0$  and complementary slackness is  $\bar{\mu}_i(G_i - K) = 0$  and  $\underline{\mu}_i G_i = 0$  for  $i = 1, 2$ .

(c) Determine the optimal rate of production of the generators and the value of all KKT multipliers. What is the interpretation of the respective KKT multipliers?

The marginal utility at the full output of the generators,  $G_1 + G_2 = 700$  MW is  $U'(700) = 8000 - 10 \cdot 700 = 1000$  €/MWh, which is higher than the costs  $c_i$ , so we'll find optimal rates  $g_1^* = G_1$  and  $g_2^* = G_2$  and  $d^* = G_1 + G_2$ . This means  $\lambda = U'(G_1 + G_2) = 1000$  €/MWh, which is the market price. Because the lower constraints on the generator output are not binding, from complementary slackness we have  $\underline{\mu}_i = 0$ . The upper constraints are binding, so  $\bar{\mu}_i \geq 0$ . From stationarity  $\bar{\mu}_i = \lambda - c_i + \underline{\mu}_i$ , which is the increase in social welfare if Generator  $i$  could increase its capacity by a marginal amount.

$$\bar{\mu}_1 = 1000 - 20 = 980 \text{ €/MWh}$$

$$\bar{\mu}_2 = 1000 - 50 = 950 \text{ €/MWh}$$

### SOLUTION IV.3 (EFFICIENT DISPATCH IN A TWO-BUS POWER SYSTEM).

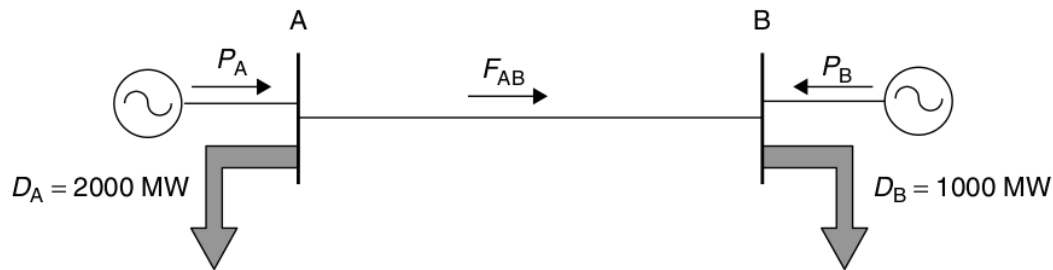


Figure 1: A simple two-bus power system.

Consider the two-bus power system shown in Figure 1, where the two nodes represent two markets, each with different total demand, and one generator at each node. At node A the demand is  $D_A = 2000$  MW, whereas at node B the demand is  $D_B = 1000$  MW. Furthermore, there is a transmission line with a capacity denoted by  $F_{AB}$ . The marginal cost of production of the generators connected to buses A and B are given respectively by the following expressions:

$$MC_A = 20 + 0.02G_A \quad \text{€/MWh}$$

$$MC_B = 15 + 0.03G_B \quad \text{€/MWh}$$

Assume that the demands  $D_A$  and  $D_B$  are constant and insensitive to price, that energy is sold at its marginal cost of production and that there are no limits on the output of the generators.

- (a) Calculate the price of electricity at each bus, the production of each generator, the flow on the line, and the value of any KKT multipliers for the following cases:

The price of electricity is the value of the dual variable at the nodal balance equation. Use the following nomenclature: price  $\lambda_{A/B}$ , generation  $G_{A/B}$ , flow  $F_{AB}$ .

- (i) The line between buses A and B is disconnected.

$$\lambda_A = 80 \text{ €/MWh}, \lambda_B = 35 \text{ €/MWh},$$

$$G_A = 2000 \text{ MW}, G_B = 1000 \text{ MW}, F_{AB} = 0$$

- (ii) The line between buses A and B is in service and has an unlimited capacity.

$$\lambda_A = 53 \text{ €/MWh}, \lambda_B = 53 \text{ €/MWh},$$

$$G_A = 1100 \text{ MW}, G_B = 1900 \text{ MW}, F_{AB} = -900 \text{ MW}$$

- (iii) The line between buses A and B is in service and has an unlimited capacity, but the maximum output of Generator B is 1500 MW.

$$\lambda_A = 65 \text{ €/MWh}, \lambda_B = 65 \text{ €/MWh},$$

$$G_A = 1500 \text{ MW}, G_B = 1500 \text{ MW}, F_{AB} = -500 \text{ MW}$$

- (iv) The line between buses A and B is in service and has an unlimited capacity, but the maximum output of Generator A is 900 MW. The output of Generator B is unlimited.

$$\lambda_A = 57 \text{ €/MWh}, \lambda_B = 57 \text{ €/MWh},$$

$$G_A = 900 \text{ MW}, G_B = 2100 \text{ MW}, F_{AB} = -1100 \text{ MW}$$

- (v) The line between buses A and B is in service but its capacity is limited to 600 MW. The output of the generators is unlimited.

$$\lambda_A = 62 \text{ €/MWh}, \lambda_B = 47 \text{ €/MWh},$$

$$G_A = 1400 \text{ MW}, G_B = 1600 \text{ MW}, F_{AB} = -600 \text{ MW}$$

- (b) Calculate the generator revenues, generator profits, consumer payments and consumer net surplus for all the cases considered in the above problem. Who benefits from the line connecting these two buses?

Generator revenues  $R_i$ , generator costs  $C_i$ , generator profits  $P_i$ , consumer payments  $E_i$ . Find the generator profits by subtracting the costs from the revenue. Costs are given by integrating the marginal cost, i.e.  $C_A = 20G_A + 0.015G_A^2$  and  $C_B = 15G_B + 0.01G_B^2$ . The generator at B and the consumers at A benefit from the line (price increases at B, decreases at B).

Case	(i)	(ii)	(iii)	(iv)	(v)
$E_A$ (€)	160,000	106,000	130,000	114,000	124,000
$R_A$ (€)	160,000	58,300	97,500	51,300	86,800
$C_A$ (€)	100,000	40,150	63,750	30,150	57,400
$P_A$ (€)	60,000	18,150	33,750	21,150	29,400
$E_B$ (€)	35,000	53,000	65,000	57,000	47,000
$R_B$ (€)	35,000	100,700	97,500	119,700	75,200
$C_B$ (€)	25,000	64,600	45,000	75,600	49,600
$P_B$ (€)	10,000	36,100	52,500	44,100	25,600

(c) Calculate the congestion surplus for case (v). For what values of the flow on the line between buses A and B is the congestion surplus equal to zero?

Congestion surplus is 9000 €:

$$(E_A + E_B) - (R_A + R_B) = |F_{AB}| \times (\lambda_A - \lambda_B)$$

Congestion surplus is equal to zero when the flow  $F_{AB} = 0$ , or when it is equal to the unconstrained value  $F_{AB} = -900$  MW (then  $\lambda_A = \lambda_B$ ).