Notes taken at the

RTG Lectures

on the subjects of

Torsion Invariants

by Prof. Dr. R. Sauer

Harmonic maps

by Dr. A. Sanders

Karlsruhe and Heidelberg

Winter 2016/17

Contents

I Torsion Invariants [Roman Sauer] 3

1 Review of Euler characteristic and Lefschetz numbers. 3

1.1 CW Complexes 3

1.2 Euler characteristic 3

1.3 Review of cellular homology 4

1.4 Lefschetz number 5

2 Whitehead torsion 6

2.1 Introduction/Motivation 6

2.2 Whitehead group and lower K-theory 8

2.3 Whitehead torsion for chain complexes 9

2.4 Whitehead torsion of maps between CW-complexes 14

II Harmonic Maps [Andy Sanders] 21

1 Basics of harmonic maps 21

1.1 Background differential geometry 21

1.2 Definition of harmonic maps of 1st variation formula 23

2 Example and the Bochner formula (a glimpse of rigidity) 24

2.1 Composition laws for harmonic maps 26

2.2 Bochner formulas 27

3 The Eells-Sampson existence theorem 28

3.1 1st short time existence 29

3.2 Need another Bochner formula 29

3.3 Moser-Harnack inequality 30

3.4 Black box # 37 30

4 Harmonic maps and Teichmüller theory 31

Chapter I

Torsion Invariants [Roman Sauer]

Torsion invariants fall into a class of so-called “secondary invariants” of topological spaces in the sense that they are only defined if a certain class of “primary invariants” (e.g. Betti numbers) vanish. Often they reveal more subtle geometric information. The following will contain a discussion of Whitehead and Reidemeister torsion. Informally, corresponding primary invariants are Lefschetz numbers (Whitehead torsion) and the Euler characteristic (Reidemeister torsion).

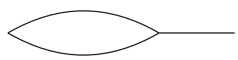
1. Review of Euler characteristic and Lefschetz numbers.

1.1. CW Complexes

Definition. A (finite) *CW-complex* is a hausdorff space with a decomposition E into (finitely many) cells (space homeomorphic to some $\mathbb{R}^n$) such that for every $e \in E$ there is a continuous map $\phi_e\colon D^n \to X$ with $\phi_e\colon \mathring{D}^n \xrightarrow{\cong} e$ and $\operatorname{Im}(\phi_e|_{S^{n-1}}) \subset \bigcup_{f\in E, \dim f \le n-1} f$.

Example. 1. Simplicial complexes, e.g. triangles, pyramids, etc.

2. But CW-complexes are more general, the following graph is CW for example:



One can even attach a disc along its boundary to a single 1-cell.

1.2. Euler characteristic

Definition. The Euler class $\chi(X)$ of a finite CW-complex X is defined as $\chi(X) = \sum_{i\geq 0}(-1)^i\#(i\text{-cells of } X) \in \mathbb{Z}$.

Theorem (Euler-Poincaré).

$$\chi(X) = \sum_{i\geq 0}(-1)^i b_i(X),$$

where $b_i(X) = \operatorname{rk}_{\mathbb{Z}} H_i(X;\mathbb{Z})$.

In particular, χ is a homotopy invariant.

"Proof". $H_i(X;\mathbb{Z}) = H_i(C_*^{CW}(X))$, where $C_*^{CW}(X)$ is the cellular chain complex

$$\cdot \to C_{i+1}^{CW}(X) \xrightarrow{\partial} \underbrace{C_i^{CW}(X)}_{\cong \mathbb{Z}^{\#i\text{-cells}}} \xrightarrow{\partial} C_{i-1}^{CW}(X) \to \cdots$$

Thus $\chi(C_*) := \sum_{i\geq 0}(-1)^i \operatorname{rk}_{\mathbb{Z}}(C_i)$ and $\chi(C^{CW}(X)) = \chi(X)$. This boils down to

$$\chi(C_*) = \sum_{i\geq 0} \operatorname{rk}_{\mathbb{Z}} H_i(C_*)(= \chi(H_*(C_*))].$$

This is just additivity of the rank! Consider

$$C_1 \xrightarrow{\partial} C_0$$

and note that we have the exact sequences $0 \to \operatorname{Im}\partial \to C_0 \to \underbrace{H_0}_{=C_0/\operatorname{Im}\partial} \to 0$ and $0 \to \underbrace{H_1}_{=\operatorname{Ker}\partial} \to C_1 \xrightarrow{\partial} \operatorname{Im}\partial \to 0$.

Thus $\chi(C_*) = \operatorname{rk}_{\mathbb{Z}} C_0 - \operatorname{rk}_{\mathbb{Z}} C_1 = \operatorname{rk}_{\mathbb{Z}} \operatorname{Im}\partial + \operatorname{rk}_{\mathbb{Z}} H_0 - \operatorname{rk}_{\mathbb{Z}} H_1 - \operatorname{rk}_{\mathbb{Z}} \operatorname{Im}\partial = \operatorname{rk} H_0 - \operatorname{rk} H_1$, which completes the "proof". □

1.3. Review of cellular homology

Let X be a CW-complex with cellular decomposition E. Then we can consider the *n-skeleton*

$$X^n := \sum_{e\in E, \dim e\leq n} e,$$

which yields a filtration $X^0 \subset X^1 \subset \cdots \subset X$ such there is a push-out diagram

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ \coprod D^n \arrow[r] & X^n \end{tikzcd}$$

One could take this as an alternative definition of a CW-complex by a filtration with the push-out property. The cells can be recovered as connected components of

$X^n \setminus X^{n-1}$.
We have

$$C_i^{CW}(X) = H_i(X^i, X^{i+1}) \xleftarrow{\cong} H_i(\coprod D^i, \coprod S^{i-1}) \cong \bigoplus H_i(D^i, S^{i-1}) \cong \bigoplus \mathbb{Z}^{\#i\text{-cells}},$$

where the first isomorphism $\leftarrow$ is given by excision. The boundary maps $C_i^{CW}(X) \xrightarrow{\partial} C_{i-1}^{CW}(X)$ come from

$$H_i(X^i, X^{i-1}) \to H_{i-1}(X^{i-1}) \to H_{i-1}(X^{i-1}, X^{i-2}).$$

Under this isomorphism, the matrix entry belonging to (e, f) where e is an n-cell, f an $(n-1)$-cell is the *degree* of the map.

$$S^{i-1} \xrightarrow{\phi_e|_{S^{n-1}}} X^{i-1} \xrightarrow{\text{proj}} X^{i-1}/(X^{i-1} \setminus f) \xleftarrow{\phi_f, \cong} D^{i-1}/S^{i-2} \cong S^{i-1}.$$

Example. Consider the torus as an identification square. We convince ourselves that the cellular chain complex is given as $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$, where $1 \mapsto (0, 0)$, since it is described by a map $S^1 \to S^1$ traversing the 2-cell according to orientation has degree 0.

1.4. Lefschetz number

Recall that a map $f\colon X \to P$ between CW-complexes is *cellular*, if $f(X^i) \subset Y^i$ for all i.

Theorem (Cellular approximation). *Any map between CW-complexes is homotopic to a cellular map.*

Definition. The *Lefschetz number* of a self-map $f\colon X \to X$ of a finite CW-complex is defined as

$$\Lambda(f) = \sum_{i \geq 0} (-1)^i \operatorname{tr} C_i^{CW}(f) \in \mathbb{Z}\,.$$

Remark. $\Lambda(\mathrm{id}_X) = \chi(X)$.

The following theorem yields a description of Lefschetz numbers by homology.

Theorem. $\Lambda(f) = \sum_{i \geq 0} (-1)^i \operatorname{tr} H_i(f)$.

Thus, this number only depends on the homotopy class of f.

Proof. Similar to the proof of Euler-Poincaré using the additivity of the trace, i.e. in the situation

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{1.1}$$

we have $\mathrm{tr}(b) = \mathrm{tr}(a) + \mathrm{tr}(c)$. □

Theorem. *If f has no fixed point, then $\Lambda(f) = 0$.*

Remark. The converse is not true (think of counterexamples, e.g. $S^1 \wedge S^1$), although there is one in the case of simply-connected closed manifolds.

Proof. Let X be metrizable and let d be a metric. If X is compact, there exists an $\varepsilon > 0$ with $d(f(x), x) > 3\varepsilon$. One can "refine" the CW-structure to a new one such that every cell has diameter $< \varepsilon$. By cellular approximation we can see that there exists a cellular map $g\colon X \to X$ with $g \simeq f$ and $d(g(x), f(x)) < \varepsilon$. Thus $g(\overline{e}) \cap \overline{e} = \emptyset$ for every cell e. Hence, the diagonal matrix entries of each $C_i^{CW}(g)$ are zero and thus $\Lambda(g) = \Lambda(f) = 0$. □

2. Whitehead torsion

2.1. Introduction/Motivation

Given a homotopy equivalence $f\colon X \xrightarrow{\simeq} Y$ of finite CW-complexes, Whitehead torsion is an assignment $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$ living in the so-called Whitehead group.

Theorem (Properties of Whitehead torsion). *(1) homotopy invariance*

(2) [1] If $f\colon X \to Y$ is a homeomorphism, then $\tau(f) = 0$.

(3) additivity: A cellular push-out is a diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, hook, "i"] & X_2 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

with X_i be CW-complexes, where f is cellular and i is an inclusion of a sub-complex. If the diagram

$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[dr, "f_0", "\simeq"'] \arrow[dd, hook] & & X_2 \arrow[dr, bend left, "f_2", "\simeq"'] \arrow[dd] & \\
& Y_0 \arrow[rr, "\phi"] \arrow[dd, hook, "j"] & & Y_2 \arrow[dd, "i"] \\
X_1 \arrow[rr] \arrow[dr, bend right, "f_1", "\simeq"'] & & X \arrow[dr, dashed, "f"] & \\
& Y_1 \arrow[rr, "\psi"] & & Y
\end{tikzcd}$$

[1] This is a deep theorem of Chapman.

is a map of cellular push-outs such that f_i are homotopy equivalences. Then f is a homotopy equivalence and

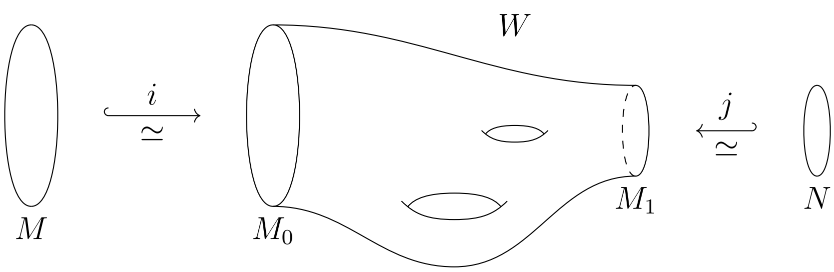
$$\begin{aligned}\tau(f) &= "\tau(f_1) + \tau(f_2) - \tau(f_0)" \\ &= \psi_*(\tau(f_1)) + j_*(\tau(f_2)) - (\psi \circ i)_*(\tau(f_0)) \in \mathrm{Wh}(\pi_1(Y)).\end{aligned}$$

A similar additivity holds for the Lefschetz number.[2]

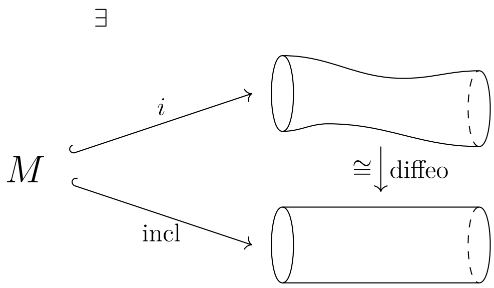
(4) composition formula. If we have $X \xrightarrow[\simeq]{f} Y \xrightarrow[\simeq]{g} Z$ *, then*

$$\begin{aligned}\tau(g \circ f) &= "\tau(f) + \tau(g)" \\ &= g_*(\tau(f) + \tau(g)) \in \mathrm{Wh}(\pi_1 Z).\end{aligned}$$

Theorem (s-cobordism theorem (Mazur, Barden, Stallings, Smale)). *Let M be a closed smooth manifold of dimension ≥ 5. Let (W, i, j) be an s-cobordism*

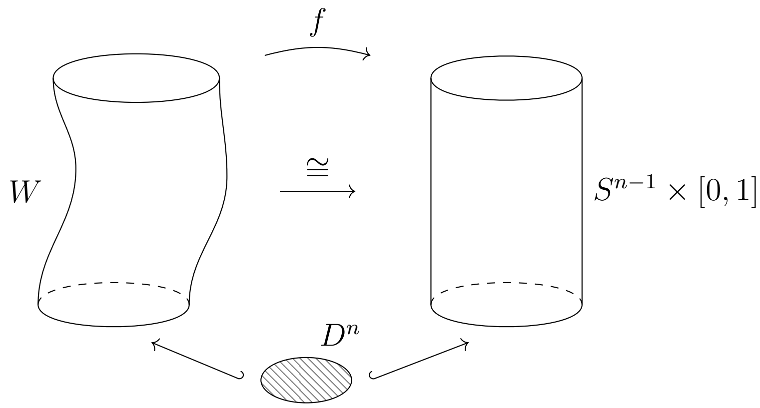


i.e. $\partial W = M_0 \coprod M_1$ and $i\colon M \hookrightarrow W$, $j\colon N \hookrightarrow W$ are homotopy equivalences. Then $\tau(M \xrightarrow{i} W) = 0$ if and only if (W, i_0, i_1) is trivial, i.e.



This theorem implies the Poincaré conjecture in dimensions at least 6, which says that if M is a closed (smooth) manifold that is homotopy equivalent to S^n, then M is homeomorphic to S^n.
The proof of this implication is basically along these lines: Pick disjoint embedded n-disks in M and remove them. The result is a manifold W with two boundary components. Consider the s-cobordism theorem for this manifold as depicted in the figure below, where f is a diffeomorphism of $(n-1)$-spheres.

[2]Idea. $0 \to C_1(X_0) \to C_*(X_1) \oplus C_*(X_2) \to C_*(X) \to 0$ exact.



By filling top and bottom, the Poincaré conjecture is implied, if we can extend a diffeomorphism $f \colon S^{n-1} \xrightarrow{\cong} S^{n-1}$ to a homeomorphism $F \colon D^n \xrightarrow{\cong} D^n$. This can be done by the so-called Alexander trick ($F(tx) = tf(x)$ for $t \in [0,1], x \in S^{n-1}$).

2.2. Whitehead group and lower K-theory

Let R be a unital ring. Then

$$K_0(R) := \langle G \mid R \rangle_{\mathrm{ab}}$$

with generators $G = \{$ isomorphism classes $[P]$ of fin. gen. projective R-modules $\}$ and relations $R = \{ [P_1] = [P_0] + [P_2]$ whenever $0 \to P_0 \to P_1 \to P_2 \to 0$ is exact $\}$. (Recall that a direct summand of in a free R-module is called a *projective module.*) This can be understood as a kind of universal dimension for projective R-modules.

$$K_1(R) := \langle G \mid R \rangle_{\mathrm{ab}}$$

with generators $G = \{$ conjugacy classes $[f]$ of automorphisms $f \colon P \to P$ of fin. gen. projective R-modules $\}$ and relations R given as follows.

(i) Every commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\cong"', "f_0"] & P_1 \arrow[r] \arrow[d, "\cong"', "f_1"] & P_2 \arrow[r] \arrow[d, "\cong"', "f_2"] & 0 \\
0 \arrow[r] & P_0 \arrow[r] & P_1 \arrow[r] & P_2 \arrow[r] & 0.
\end{tikzcd}$$

gives rise to a relation $[f_1] = [f_0] + [f_2]$.

(ii) $f, g \colon P \xrightarrow{\cong} P$ yield a relation $[f \circ g] = [f] + [g]$.

This can be understood as an attempt to define a universal determinant of an automorphism.

There is a more common definition of $K_1(R)$ in terms of the general linear groups with coefficients in R. Recall that $\mathrm{GL}(R) = \mathrm{colim}_{n\to\infty} \mathrm{GL}_n(R)$ with respect to the inclusion $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ to the upper left block. Then

$$K_1(R) = \mathrm{GL}(R)_{\mathrm{ab}} = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)].$$

The so-called *Whitehead lemma* states that $[\mathrm{GL}(R), \mathrm{GL}(R)] = E(R)$, where $E(R)$ is the subgroup of $\mathrm{GL}(R)$ generated by all elementary upper triangular matrices with ones on the diagonal. As a consequence, if R is a field then the determinant defines an isomorphism $\det\colon K_1(R) \xrightarrow{\cong} \mathbb{R}\setminus\{0\}$.

To see the equivalence of these two definitions, we can use the map

$$\begin{aligned} \mathrm{GL}(R) &\to K_1(R) \\ A &\mapsto [R^n \to R^n,\ x \mapsto Ax] \end{aligned}$$

and the fact that is descents to $\mathrm{GL}(R)_{\mathrm{ab}} \to K_1(R)$.

The inverse homomorphism is given by $R^n \cong \{P \oplus Q \xrightarrow{f\otimes \mathrm{id}} P \oplus Q \mid f \text{ iso } \}$.

Definition. Let Γ be a group. Then the *Whitehead group* is defined as

$$\mathrm{Wh}(\Gamma) := \mathrm{coker}(\Gamma \times \{\pm 1\} \to K_1(\mathbb{Z}[\Gamma]),\ (\gamma, \pm 1) \mapsto \pm[\gamma])$$

Example. The Whitehead group $\mathrm{Wh}(\{1\})$ is trivial, since the determinant yields an isomorphism $K_1(\mathbb{Z}) \xrightarrow{\det} \{\pm 1\}$.

It is a conjecture that torsion-free groups Γ have vanishing Whitehead group.

Assertion: $\mathrm{Wh}(\mathbb{Z}/5) \cong \mathbb{Z}$. Here only prove that $\mathrm{Wh}(\mathbb{Z}/5)$ is infinite. We have a map $\phi_*\colon K_1(\mathbb{Z}[\mathbb{Z}/5]) \to K_1(\mathbb{C})$ induced by $\mathbb{Z}[\mathbb{Z}/5] \xrightarrow{\phi} \mathbb{C},\ t \mapsto \xi$, where $\mathbb{Z}/5 \cong \langle t\rangle$ and $\xi = \exp(2\pi i/5) \in \mathbb{C}$. Thus

$$\begin{tikzcd} K_1(\mathbb{Z}[\mathbb{Z}/5]) \arrow[r, "\phi_*"] \arrow[d] & K_1(\mathbb{C}) \arrow[r, "\det"] & \mathbb{C}^\times \arrow[r, "|-|"] & \mathbb{R}_{>0} \\ \mathrm{Wh}(\mathbb{Z}/5) \arrow[urrr, bend right, "\tau"] & & & \end{tikzcd}$$

One can see that $1 - t - t^{-1}$ is a unit in $\mathbb{Z}[\mathbb{Z}/5]$, since $(1 - t - t^{-1})(-t^2 - t^3) = 1$ and thus $\tau([1 - t - t^{-1}]) \neq 1$

2.3. Whitehead torsion for chain complexes

In the following let us repeat some preliminaries on chain complexes. Let R be a (not necessarily commutative) ring.

Let $f_*\colon C_* \to D_*$ be an R-chain map. The *mapping cylinder* $\mathrm{cyl}(f_*)$ is an R-chain complex with p-th differential

$$C_p \oplus C_{p-1} \oplus D_p \xrightarrow{\begin{pmatrix} c_p & -\mathrm{id} & 0 \\ 0 & -c_{p-1} & 0 \\ 0 & f_{p-1} & f \end{pmatrix}} C_{p-1} \oplus C_{p-2} \oplus D_{p-1},$$

Remark. For a continuous map $f\colon X \to Y$ we have the topological mapping cylinder.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_0"] & Y \arrow[d, hook] \\ X \times [0,1] \arrow[r] & \mathrm{cyl}(f) \end{tikzcd}$$

If X, Y are CW-complexes and f is cellular, then

$$\mathrm{cyl}(C_*(f)\colon C_*(X) \to C_*(Y)) = C_*(\mathrm{cyl}(f)).$$

The *mapping cone* $\mathrm{cone}(f_*\colon C_* \to D_*)$ is a quotient of $\mathrm{cyl}(f_*)$ by the obvious copy of C_*, so its differential is

$$C_{p-1} \oplus D_+ \xrightarrow{\begin{pmatrix} -c_{p-1} & 0 \\ f_{p-1} & d_p \end{pmatrix}} C_{p-2} \oplus D_{p-1}$$

Remark. Again there is a topological analogue, the topological mapping cone $\mathrm{cone}(f) = \mathrm{cyl}(f)/X \times \{1\}$. These are related via

$$\mathrm{cone}_i(C_*(f)) = C_i(\mathrm{cone}(f)) \text{ for } i > 0.$$

The *suspension* ΣC_* of an R-chain complex C_* is a chain complex with p-th differential

$$C_{p-1} \xrightarrow{-c_{p-1}} C_{p-2},$$

which is isomorphic to a quotient of $\mathrm{cone}(\mathrm{id}_{C_*})$ by C_*.
We have two exact sequences

$$0 \to C_* \to \mathrm{cyl}(f_*) \to \mathrm{cone}(f_*) \to 0 \qquad 0 \to D_* \to \mathrm{cone}(f_*) \to \Sigma C_* \to 0$$

Definition. An R-chain complex C_* is *finite*, if $|C_p| = 0$ for $p >> 0$ and each C_p is finitely generated. It is called *projective*, if each C_p is projective; *free*, if each C_p is free, and *based free*, if each C_p is based free with a preferred basis.

Remark. Let $f_*\colon C_* \to D_*$ be a chain map between projective chain complexes. Then the following statements are equivalent

1. f_* is a homology isomorphism (i.e. $H_i(f_*)$ is an isomorphism for all i)

2. f_* is a chain homotopy equivalence

3. $\operatorname{cone}(f_*)$ is contractible (i.e. $\operatorname{cone}(f_*) \simeq 0$).

This can be seen from the following sequence (together with the fundamental lemma in homological algebra to show the equivalence of the first two statements).

$$\begin{tikzcd} 0 \arrow[r] & C_* \arrow[r] & \operatorname{cyl}(f) \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & \operatorname{cone}(f_*) \arrow[r] & 0 \\ & & D_* & & \end{tikzcd}$$

A short exact sequence of chain complexes induces a long exact sequence in homology associated to C_*, which implies $\operatorname{cone}(f_*) = 0 \rightsquigarrow H_*(\operatorname{cone}(f_*)) = 0 \rightsquigarrow f_*$ is homology isomorphism.

Lemma. *Let C_+ be a based free, finite R-chain complex that is contractible. Let $\gamma_p \colon C_p \to C_{p+1}$ for $p \in \mathbb{Z}$ be a chain contraction, i.e.*

$$c_{p+1} \circ \gamma_p + \gamma_{p-1} \circ c_p = \operatorname{id} - 0.$$

Then the R-homomorphism $(c_* + \gamma_*) \colon C_{odd} \to C_{ev}$ (where $C_{odd} = \bigoplus_p C_{2p+1}$ and $C_{ev} = \bigoplus_p C_{2p}$) is an isomorphism. Let A be its representing matrix. Its class $[A] \in K_1(R)$ is independent of the choice of γ_*.

Example. Let $C_p = 0$ unless $i \in \{0, 1, 2\}$. Then

$$\begin{tikzcd} 0 \arrow[r] & C_2 \arrow[r, "c_2"'] & C_1 \arrow[l, bend right, "\gamma_1"'] \arrow[r, "c_1"'] & C_0 \arrow[l, bend right, "\gamma_0"'] \arrow[r] & 0 \end{tikzcd}$$

is the full complex and thus "contractible" means "short exact". $C_1 \xrightarrow{\cong} C_0 \oplus C_2$ via $x \mapsto c_1(x) + \gamma_1(x)$ with inverse $C_0 \oplus C_2 \xrightarrow{\cong} C_1, (x, y) \mapsto \gamma_0(x) + c_2(y)$.
Let $\tilde{\gamma}$ be another chain contraction

$$C_0 \oplus C_2 \xrightarrow[(\tilde{\gamma}_1, c_2)]{\cong} C_1 \xrightarrow[(c_1, \gamma_1)]{\cong} C_0 \oplus C_2$$

Let $x + y \in C_0 \oplus C_2$. Then

$$\begin{aligned} \tilde{\gamma}_0(x) + c_2(y) \mapsto c_1\tilde{\gamma}_0(x) &+ \gamma_1\tilde{\gamma}_0(x) + \gamma_1 c_2(y) \\ &= (x - \underbrace{\tilde{\gamma}_2 c_0(x))}_{=0} + \gamma_1\tilde{\gamma}_0(x) + (y - \underbrace{c_3\gamma_2(y))}_{=0} \\ &= x + y + \gamma_1\tilde{\gamma}_0(x), \end{aligned}$$

which is represented by a matrix $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Proof. Let γ_*, δ_* be two chain contractions of C_*. Then we consider

$$(c_* + \delta_*)_{\mathrm{odd}} \colon C_{\mathrm{odd}} \xrightarrow{A} C_{\mathrm{ev}} \text{ and } (c_* + \delta_*)_{\mathrm{ev}} \colon C_{\mathrm{ev}} \xrightarrow{B} C_{\mathrm{odd}}$$

represented by matrices A and B. Define $\mu_n = (\gamma_{n+1} - \delta_{n+1}) \circ \delta_n$ and $\nu_n = (\delta_{n+1} - \gamma_{n+1}) \circ \gamma_n$. Then $(\mathrm{id} + \mu_*)_{\mathrm{odd}}$, $(\mathrm{id} + \nu_*)_{\mathrm{ev}}$ and $(c_* + \gamma_*)_{\mathrm{odd}} \circ (\mathrm{id} + \mu_*)_{\mathrm{odd}} \circ (c_* + \delta_*)_{\mathrm{ev}}$ are represented by upper triangular matrices with ones on the diagonal.

Thus $[A] = -[B]$ in $K_1(R)$ and B is independent of the choice of γ_*, hence A is independent of the choice of γ_*. $\square$

Definition. (i) For a contractible, based free, finite R-chain complex C_* define

$$\tau(C_*) := [(c_* + \gamma_*)_{\mathrm{odd}}] \in K_1(R)$$

(for some or every) choice of $\gamma_* \colon C_* \simeq 0$).

(ii) Let $f_* \colon C_* \to D_*$ be a chain homotopy equivalence of based free, finite R-chain complexes. The *Whitehead torsion* of f_* is

$$\tau(f_*) := \tau(\mathrm{cone}(f_*)) \in K_1(R).$$

We say that a short exact sequence of based free modules

$$0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$$

is *based exact*, if $\mathrm{basis}_B = B_1 \coprod B_2$, such that $B_1 = j(\mathrm{basis}_{A)}$ and $p(B_2) = \mathrm{basis}_C$.

Lemma. *Consider the following diagram with based exact rows.*

$$\begin{tikzcd}
0 \arrow[r] & C'_* \arrow[r] \arrow[d, "\simeq"', "f_*"] & D'_* \arrow[r] \arrow[d, "\simeq"', "g_*"] & E'_* \arrow[r] \arrow[d, "\simeq"', "h_*"] & 0 \\
0 \arrow[r] & C_* \arrow[r] & D_* \arrow[r] & E_* \arrow[r] & 0
\end{tikzcd}$$

Then $\tau(g_*) = \tau(f_*) + \tau(h_*)$.

Proof. We have the following diagrams with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C'_* \arrow[r] \arrow[d, "\simeq"] & D'_* \arrow[r] \arrow[d, "\simeq"] & E'_* \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \operatorname{cyl}(f_*) \arrow[r] \arrow[d] & \operatorname{cyl}(g_*) \arrow[r] \arrow[d] & \operatorname{cyl}(h_*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{cone}(f_*) \arrow[r] \arrow[d] & \operatorname{cone}(g_*) \arrow[r] \arrow[d] & \operatorname{cone}(h_*) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We may assume that the short exact sequence given by the columns

$$0 \to C_* \xrightarrow{j_*} D_* \xrightarrow{p_*} E_* \to 0 \tag{2.1}$$

is a based exact sequence of contractible based exact sequence of contractible based free, finite chain complexes and then have to prove that $\tau(D_*) = \tau(C_*) + \tau(E_*)$.
The sequence (2.1) splits as chain complexes.[3] Let e_* be a contraction of E_* and let $\sigma_i\colon E_i \to D_i$ be a split of p_i for all i. Set $s_i\colon E_i \to D_i$, $s_i := d_{i+1}\circ\sigma_{i+1}\circ\varepsilon_i + \sigma_i\circ\varepsilon_{i-1}\circ e_i$.
Claim: s_* is a chain map and $p_* \circ s_* = \mathrm{id}_{E_*}$.
Hence we obtain an isomorphism of chain complexes

$$(j_*, s_*)\colon C_* \oplus E_* \xrightarrow{\cong} D_*,$$

which has a corresponding matrix of the form $\begin{pmatrix} \mathrm{Id} & * \\ 0 & \mathrm{Id} \end{pmatrix}$.
We can finish the proof with the following remark. If $u_*\colon C_* \to D_*$ is a chain isomorphism of then

$$\tau(C_*) - \tau(D_*) = \sum_p (-1)^p [u_p] \in K_1(R).$$

This can be shown by transporting a chain contraction γ_* for C_* to one for D_* via u_*, which yields a diagram:

$$\begin{tikzcd}
C_{\mathrm{odd}} \arrow[r] \arrow[d, "\simeq"', "u_{\mathrm{odd}}"] & C_{\mathrm{ev}} \arrow[d, "\simeq"', "u_{\mathrm{ev}}"] \\
D_{\mathrm{odd}} \arrow[r] & D_{\mathrm{ev}}.
\end{tikzcd}$$

□

Recall that last time we proved the following Lemma.

[3] This heavily depends on contractibility and is not true for arbitrary chain complexes.

Lemma. *(1) Additivity*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* & \longrightarrow & 0 \\ & & \simeq\Big\downarrow f_* & & \simeq\Big\downarrow g_* & & \simeq\Big\downarrow h_* & & \\ 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & 0 \end{array}$$

Then $\tau(g_*) = \tau(f_*) + \tau(h_*)$.

(2) Homotopy invariance. If $f_* \simeq g_* \colon C_* \xrightarrow{\simeq} D_*$*, then* $\tau(f_*) = \tau(g_*)$.

(3) Composition formula. $\tau(g_* \circ f_*) = \tau(g_*) + \tau(f_*)$.

Proof. ad (3) If $h_* \colon f_* \simeq g_*$, then we have an isomorphism

$$\mathrm{cone}(f_*) \colon C_{*-1} \oplus D_* \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ h_{*-1} & \mathrm{id} \end{pmatrix}} C_{*-1} \oplus D_* = \mathrm{cone}(g_*).$$

Thus

$$\begin{aligned} \tau(f_*) - \tau(g_*) &= \tau(\mathrm{cone}(f_*)) - \tau(\mathrm{cone}(g_*)) \\ &= \sum_{p \geq 0} (-1)^p \begin{pmatrix} \mathrm{id} & 0 \\ h_{*-1} & \mathrm{id} \end{pmatrix} = 0 \in K_1(R). \end{aligned}$$ □

2.4. Whitehead torsion of maps between CW-complexes

Let X, Y be connected finite CW-complexes and let $f \colon X \xrightarrow{\simeq} Y$ be a homotopy equivalence. Further pick base points $x \in X, y = f(x) \in Y$ and set $\pi := \pi_1(Y, y)$. We identify $\pi_1(X, x)$ with π via $\pi_1(f)$ and considering the universal covers we obtain the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow[\simeq]{\tilde{f}} & \tilde{Y} \\ \Big\downarrow \mathrm{pr}_X & & \Big\downarrow \mathrm{pr}_Y \\ X & \xrightarrow[\simeq]{f} & Y \end{array}$$

There is a unique lift $\tilde{f}$ with $\tilde{f}(\tilde{x}) = \tilde{y}$. Further, $\tilde{f}$ is π-equivariant. $\tilde{X}$ caries a CW structure $\tilde{X}^n = \mathrm{pr}_X^{-1}(X^n)$.
Thus $C_*^{\mathrm{CW}}(\tilde{f}) \colon C_*^{\mathrm{CW}}(\tilde{X}) \xrightarrow{\simeq} C_*^{\mathrm{CW}}(\tilde{Y})$, where $C_n^{CW}(\tilde{X}) = H_n(\tilde{X}^n, \tilde{X}^{n-1})$, are chain homotopy equivalences of $\mathbb{Z}\pi$-chain complexes. We obtain a $\mathbb{Z}\pi$-basis of $C_n^{\mathrm{CW}}(\tilde{X})$

by choosing (π-)push-outs:

$$\begin{array}{ccc} \coprod_{I_n} \pi \times S^{n-1} & \longrightarrow & \tilde{X}^{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{I_n} \pi \times D^n & \longrightarrow & \tilde{X}^n \end{array}.$$

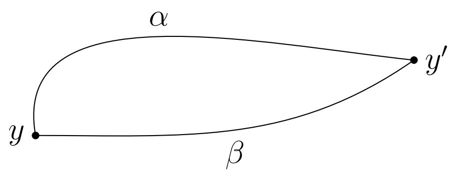
This yields a cellular basis

$$C_n^{\mathrm{CW}}(\tilde{X}) = H_n(\tilde{X}^n, \tilde{X}^{n-1}) \xleftarrow{\cong} \bigoplus_{I_n} H_n(\pi \times (D^n, S^{n-1})) = \bigoplus_{I_n} \mathbb{Z}\pi$$

The matrix of base change when we take another push-out looks like

$$\bigoplus_{I_n} \mathbb{Z}\pi \xrightarrow{\begin{pmatrix} \pm g_1 & & \\ & \ddots & \\ & & \pm g_n \end{pmatrix}_P} \bigoplus_{I_n} \mathbb{Z}\pi,$$

where P is a permutation matrix. Hence we obtain a well-defined element $\tau(C_*^{\mathrm{CW}}(\tilde{f})) \in \mathrm{Wh}(\pi)$ independent of the choices of cellular bases of $\tilde{X}, \tilde{Y}$.
It may still depend on the choice of basepoints. Let $y, y' \in Y$ be connected by two paths α and β as in the figure below.



Then $\pi_1(Y, y) \xrightarrow{\alpha_*, \beta_*} \pi_1(Y, y')$ differ by an inner automorphism of $\pi_1(Y, y')$, which induces the identity on $\mathrm{Wh}(\pi_1(Y, y'))$. So there is a canonical isomorphism

$$\phi_{y,y'} \colon \mathrm{Wh}(\pi_1(Y, y)) \xrightarrow{\cong} \mathrm{Wh}(\pi_1(Y, y'))$$

induced by any choice of path from y to y'. This yields a basepoint free definition of the Whitehead group.

Definition. $\mathrm{Wh}(\pi_1(Y)) := \operatorname{colim}_{y \in Y} \mathrm{Wh}(\pi_1(Y, y)) = \coprod_{y \in Y} \mathrm{Wh}(\pi_1(Y, y)) / \sim$ for $z \sim \phi_{y,y'}$.

One verifies that $\tau(C_*^{\mathrm{CW}}(\tilde{f})) \in \mathrm{Wh}(\pi(Y))$ is independent of all choices of basepoints.

Definition. Let $X \xrightarrow{f} Y$ be a homotopy equivalence of finite CW-complexes. Define

$$\mathrm{Wh}(\pi(Y)) := \bigoplus_{C \in \pi_0(Y)} \mathrm{Wh}(\pi(C))$$

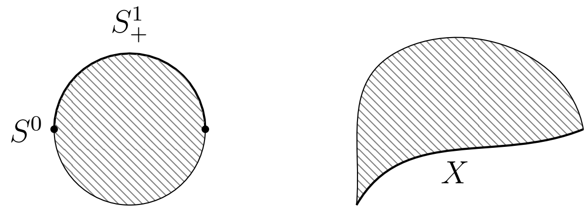
and

$$\tau(f) := \Big(\tau(C_*^{\mathrm{CW}}(\tilde{f}\colon \widetilde{f^{-1}(C)} \to \tilde{C}))\Big)_{C\in\pi_0(Y)},$$

which is called the *Whitehead torsion* of f.

Remark. The elementary properties of $\tau(f)$ stated in the beginning of chapter 2 are now direct consequences of the lemma in section 2.3.

In the following we will discuss the topological meaning of $\tau(f)$. For this let $S^{n-2} \subset S^{n-1}_+ \subset S^{n-1} \subset D^n$, where S^{n-1}_+ denotes the upper hemisphere.



$$\begin{tikzcd}
S^{n-1}_+ \arrow[r, "q"] \arrow[d, hook, "\simeq"] & X \arrow[d, hook, "\simeq"] \\
D^n \arrow[r, "\bar{q}"] & Y
\end{tikzcd}$$

Such a homotopy equivalence $X \to Y$ as in the diagram is called *elementary expansion.*
A map $r\colon Y \to X$ such that $r \circ j = \mathrm{id}_X$ is called *elementary collapse* (r will be a homotopy inverse of j).

Remark. If X is a CW-complex and $q(S^{n-2}) \subset X^{n-2}$ with $q(S^{n-1}_+) \subset X^{n-1}$, then Y inherits a natural CW-structure. Then Y is the result of attaching an $(n-1)$-cell and then an n-cell to X.

Definition. Let $f\colon X \to Y$ be a map of finite CW-complexes. We call f a *simple homotopy equivalence*, if it is homotopic to a zig-zag of elementary expansions and collapses, i.e. there exists maps $f(i)$ such that

$$\begin{tikzcd}
X = X(0) \arrow[r, "f(0)"] \arrow[rrr, bend right, "f"] & X(1) \arrow[r] & \cdots \arrow[r, "f(n-1)"] & X(n) = Y
\end{tikzcd},$$

commutes up to homotopy, where each $f(i)$ is an elementary expansion or collapse.

Theorem. *(1) A homotopy equivalence $f\colon X \to Y$ is simple if and only if $\tau(f) = 0$.*

(2) *For every element* $a \in \mathrm{Wh}(\pi(X))$ *there is a homotopy equivalence* $X \xrightarrow{f} Y$ *such that* $f_*^{-1}(\tau(f)) \in \mathrm{Wh}(\pi(X))$.

Proof. ad (1). We will only prove the direction "simple $\implies \tau(f) = 0$". We may assume that f is an elementary expansion

$$\begin{tikzcd}
0 \arrow[r] & C_*(\tilde{X}) \arrow[r, "f_*"] & C_*(\tilde{Y}) \arrow[r] & C_*(\tilde{Y}, \tilde{X}) \arrow[r] & 0 \\
0 \arrow[r] & C_*(\tilde{X}) \arrow[r, "\mathrm{id}"] \arrow[u, "\mathrm{id}"] & C_*(\tilde{X}) \arrow[r] \arrow[u, "f_*"] & 0 \arrow[r] \arrow[u, "0"] & 0
\end{tikzcd}$$

Thus, by additivity, we have $\tau(f) = \tau(C_*(\tilde{Y}, \tilde{X}))$. But this is a very simple CW-complex with differential $\mathrm{id}\colon \mathbb{Z}\pi \to \mathbb{Z}\pi$ (for $\pi = \pi_1(Y)$) from degree n to degree $n-1$ and thus $\tau(C_*(\tilde{Y}, \tilde{X})) = 0$.

The converse is more complicated.

ad (2). Let $A \in \mathrm{GL}_n(\mathbb{Z}\pi)$ ($n \geq 3$) be an element representing $a \in \mathrm{Wh}(\pi(X))$ and let $X' = X \vee \bigvee_{j=1}^{n} S^{n-1}$, where we denote the j-th inclusion of S^{n-1} into the wedge by b_j. We attach n n-cells to X' via attaching maps $f_j\colon S^{n-1} \to X'$ (relevant: $[f_j] \in ß_{n-1}(X')$ which yields Y.

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = A \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \ni \bigoplus_{j=1}^{n} \pi_{n-1}(X') \text{ for } b_j \in \pi_{n-1}(X') \curvearrowleft \pi_1(X') = \pi_1(X) = \pi.$$

One sees by closer inspection that the relative chain complex $C_*^{\mathrm{CW}}(\tilde{Y}, \tilde{X})$ has only two non-trivial chain groups in two consecutive dimensions, with boundary map realized by the $\mathbb{Z}\pi$-isomorpism A. Thus, $C_*^{\mathrm{CW}}(\tilde{Y}, \tilde{X})$ is acyclic, and since $\pi_1(\tilde{Y}, \tilde{X}) = 0$, we get from the *relative Hurewicz theorem* that $\pi_k(\tilde{Y}, \tilde{X}) = 0$ for all $k \in \mathbb{N}$. Since $n \geq 3$, we have $\pi_1(Y, X) = 0$, so we can conclude that $\pi_k(X) \cong \pi_k(Y)$ for all k, where the isomorphism is realised by the map $\pi_k(X \hookrightarrow Y)$. By *Whitehead's theorem*, $X \hookrightarrow Y$ is a homotopy equivalence, so we can indeed compute its Whitehead torsion. We get:

$$\tau(\tilde{X} \hookrightarrow \tilde{Y}) = \tau(C_*^{\mathrm{CW}}(\tilde{Y}, \tilde{X})) = \tau\left(\bigoplus_{j=1}^{n} \mathbb{Z}\pi \xrightarrow{A} \bigoplus_{j=1}^{n} \mathbb{Z}\pi\right) = a$$

for $[A] \in \mathrm{Wh}(\pi)$. $\square$

The reverse statement "$\tau(f) = 0 \implies f$ simple" follows from a geometric description of the Whitehead group. To this end, we need some lemmas about elementary collapsed and expansions.

Notation. Write

- $X \nearrow Y$, if there is a sequence of elementary expansions from X to Y.
- $X \searrow Y$, if there is a sequence of elementary collapses from X to Y.
- $X \mathrel{\nearrow\!\!\!\searrow} Y$, if there is a sequence of elementary expansions or collapses from X to Y.

Lemma (The relativity principle.). *Given cellular pushouts*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \\ Z \arrow[r] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \\ Z' \arrow[r] & W' \end{tikzcd}$$

where we assume that $Z \mathrel{\nearrow\!\!\!\searrow} Z'$ rel X. Then $W \mathrel{\nearrow\!\!\!\searrow} W'$ rel Y (by the "same" sequence of elementary expansions or collapses).

Lemma (The cylinder lemma). *Let $f\colon X \to Y$ be cellular and let $A \subset X$ be a subcomplex. Then the inclusion $\mathrm{cyl}(f|_A) \hookrightarrow \mathrm{cyl}(f)$ is a composition of elementary expansions.*
(Special case: $A = \emptyset, Y \hookrightarrow \mathrm{cyl}(f)$.)

Proof. First consider the case $X = D^n \cup_q A$ for $q\colon S^{n-1} \to A$, i.e. X is obtained by gluing an n-ball to A. This yields a pushout

$$\begin{tikzcd} S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\} \arrow[r] \arrow[d, hook, "\simeq"'] & \mathrm{cyl}(f|_A) \arrow[d, hook, "\simeq"'] \\ D^n \times [0,1] \arrow[r] & \mathrm{cyl}(f). \end{tikzcd}$$

For the left hand side of the diagram, we have $(D^n \times [0,1], S^{n-1} \times [0,1] \cup_{S^{n-1}\times\{0\}} D^n \times \{0\}) \cong (D^{n+1}, S^n_+)$, where S^n_+ is a hemisphere. Thus the pushout describes one elementar expansion. $\square$

Lemma (Relative isomorphism lemma). *If we have*

$$\begin{tikzcd} & Y_1 \arrow[dd, "h"', "\cong"] \\ X \arrow[ur, hook] \arrow[dr, hook] & \\ & Y_2 \end{tikzcd},$$

where h is a CW isomorphism, then $Y_1 \mathrel{\nearrow\!\!\!\searrow} Y_2$ rel X.

Proof. By the cylinder lemma we have $X \times [0,1] \cup Y_2 \times \{1\} = \mathrm{cyl}(h|_X) \nearrow \mathrm{cyl}(h)$. By the same proof, $X \times [0,1] \cup Y_1 \times \{0\} \nearrow \mathrm{cyl}(h)$.
Applying the relativity principle to $\mathrm{pr}\colon X \times [0,1] \to X$

$$\begin{tikzcd}
X \times [0,1] \arrow[r, "\mathrm{pr}"] \arrow[d, hook] & X \arrow[d, hook] & X \times [0,1] \arrow[r, "\mathrm{pr}"] \arrow[d, hook] & X \arrow[d, hook] \\
\mathrm{cyl}(h|_X) \arrow[r] \arrow[rr, bend right, "\nearrow\!\!\!\searrow"'] & Y \arrow[rr, dashed, bend right, "\nearrow\!\!\!\searrow"'] & X \times [0,1] \cup Y_1 \times \{0\} \arrow[r] & Y
\end{tikzcd}$$

$\square$

Lemma (Homotopy lemma). *Let cellular maps $f, g\colon K \to L$ be homotopic. Then $\mathrm{cyl}(f) \nearrow\!\!\!\searrow \mathrm{cyl}(g)$ rel $K \cup L$ (top, bottom).*

Proof. Let H be a cellular homotopy with $H_0 = f, H_1 = g$. We have to show that

$$\mathrm{cyl}(H_0) \cup K \times [0,1] \nearrow \mathrm{cyl}(H) \nwarrow \mathrm{cyl}(H_1) \cup K \times [0,1].$$

This is implied by the general fact for $X = K \times [0,1]$ and $X_0 = K \times \{0\}$ and $f = H$.

($*$) If $f\colon X \to Y$ is cellular and $X \supset X_0 \nearrow X$, then $X \cup \mathrm{cyl}(f|_{X_0}) \nearrow \mathrm{cyl}(f)$.

Now apply the relativity principle with respect to $\mathrm{pr}\colon K \times [0,1] \to K$

$$\begin{tikzcd}
K \times [0,1] \arrow[r, "\mathrm{pr}"] \arrow[d, hook] & K \arrow[d] & K \times [0,1] \arrow[r, "\mathrm{pr}"] \arrow[d, hook] & K \arrow[d, hook] \\
\mathrm{cyl}(H_0) \cup K \times [0,1] \arrow[r] \arrow[rr, bend right, "\nearrow\!\!\!\searrow"'] & \mathrm{cyl}(H_0) & \mathrm{cyl}(H_1) \cup K \times [0,1] \arrow[r] & \mathrm{cyl}(H_1)
\end{tikzcd}$$

Thus, $\mathrm{cyl}(f) \nearrow\!\!\!\searrow \mathrm{cyl}(g)$. $\square$

Let (Y,X) and (Z,X) be paris of finite CW complexes such that $X \overset{\simeq}{\hookrightarrow} Y$ and $X \overset{\simeq}{\hookrightarrow} Z$ are homotopy equivalences. We say that (Y,X) and (Z,X) are *equivalent*, if $Y \nearrow\!\!\!\searrow X$ rel X.

Definition. The *geometric Whitehead group* $\mathrm{Wh}^{\mathrm{geo}}(X)$ of X is the set of equivalence classes of such pairs (Y,X) of finite CW complexes with $X \overset{\simeq}{\hookrightarrow} Y$.
$\mathrm{Wh}^{\mathrm{geo}}(X)$ carries the structure of an abelian group.

1. Abelian addition: $[Y,X] + [Z,X] = [Y \cup_X Z, X]$ This is well-defined since for

$(Y', X) \sim (Y, X)$

$$\begin{tikzcd}
X \arrow[r, hook, "\simeq"] \arrow[d, hook, "\simeq"] & Z \arrow[d, hook, "\simeq"] & X \arrow[r, "\simeq"] \arrow[d, hook, "\simeq"] & Z \arrow[d, hook] \\
Y \arrow[r, hook] \arrow[rr, bend right, "\nearrow\!\searrow"'] & Y \cup_X Z & Y' \arrow[r] & Y' \cup_X Z
\end{tikzcd}$$

Thus, $(Y \cup_X Z, X) \sim (Y' \cup_X Z, X)$.

2. Zero element: $[X, X]$.

3. Inverse: Take a pair (Y, X) and let $D \colon Y \to X$ be a cellular strong deformation retract.

Figure I.1: TODO: Add figure for $2\,\mathrm{cyl}(D)$.

Claim: $[2\,\mathrm{cyl}(D), X] = -[Y, X]$.

$$[2\,\mathrm{cyl}(D), X] + [Y, X] = [2\,\mathrm{cyl}(D) \cup_X Y, X] = [\mathrm{cyl}(i \circ D) \cup \widetilde{\mathrm{cyl}}(D), X]$$

for $i \circ D \colon Y \to Y \simeq \mathrm{id}$, the homotopy lemma yields $\mathrm{cyl}(i \circ D) \nearrow\!\searrow Y \times [0,1]$ rel $Y \times \{0\} \cup Y$

$$= [Y \times [0,1], \widetilde{\mathrm{cyl}}(D), X]$$

$Y \times [0,1] \searrow X \times [0,1] \cup Y \times \{0\}$ which follows from $(*)$ for id_Y.

$$= [Y \times [0,1] \cup \widetilde{\mathrm{cyl}}(D), X].$$

$\widetilde{\mathrm{cyl}}(D) \searrow X \times [-1, 0]$ by the cylinder lemma for D.

$$= [X \times [-1,1], X] = [X, X] = 0$$

Theorem.

$$\mathrm{Wh}^{geo}(X) \xrightarrow{\cong} \mathrm{Wh}(\pi(X)), \quad [Y, X] \mapsto i_*^{-1}(\tau(i \colon X \hookrightarrow Y))$$

is an isomorphism of abelian groups.

In some sense this is a topological version of the s-cobordism theorem.

Chapter II

Harmonic Maps [Andy Sanders]

Also consider the notes `www.mathi.uni-heidelberg.de/~asanders/harmonicmaps.htm`.

1. Basics of harmonic maps

In the following let every manifold be oriented (for integration safety reasons).

1.1. Background differential geometry

Let $E \to M$ be an $\mathbb{R}$-vector bundle over M (second countable, hausdorff manifold) of rank r. A *connection* ∇ on E is an $\mathbb{R}$-linear map

$$\nabla \colon \Omega^0(E) \to \Omega^0(\mathrm{T}^* M \otimes_{\mathbb{R}} E) =: \Omega^1(M, E), s \mapsto \nabla_{-} s$$

where $\Omega^0(E)$ denotes smooth sections in E, such that

1. $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$,
2. $\nabla_X (s + s') = \nabla_X s + \nabla_X s'$
3. $\nabla_{fX} s = f \nabla_X s$
4. $\nabla_X (fs) = f \nabla_X s + X(f) s$.

Let q be an inner product on E. We say that ∇ is a *metric connection* for q, if for all $s, t \in \Omega^0(E)$ we have

$$\mathrm{d}q(s, t) = q(\nabla s, t) + q(s, \nabla t).$$

Example. Let (M, g) be a riemannian manifold with tangent bundle $E = \mathrm{T}\, M$ and Levi-Civita connection ∇ of g.

Let $X, Y \in \Omega^0(M)$ be vector fields, i.e. $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ in local co-ordinates. (Abbreviate ∂_i for $\frac{\partial}{\partial x^i}$.)

$$\nabla_X Y = \nabla_{X^i \partial_i} Y^i \partial_i = X^i(\nabla_{\partial_i} Y^i \partial_i) = X^i(\partial_i Y^i \partial_i + Y^i \nabla_{\partial_i} \partial_i) = X^i(\partial_i Y^i \partial_i + Y^i \Gamma^k_{ij} \partial_i)$$

where $\Gamma^k_{ij} = g^{km}(\partial_i g_{im} + \partial_j g_{im} - \partial_m g_{ij})$ for $g_{ij} = g(\partial_i, \partial_j)$ and g^{km} is the km-entry of g^{-1}.

Out of E one can build another bundle $E^* = \operatorname{Hom}(E, \mathbb{R})$ and given another vector bundle F, one can build $\operatorname{Hom}(E, F)$,

Definition. Let $(E, \nabla) \to M$ be a vector bundle with a connection over M. The space of *p-forms* on m with values in E is the $C^\infty(M)$-module $\Omega^p(M, E) = \Omega^0(M, \bigwedge^p \mathrm{T}^* M \otimes E)$. Elements α in $\Omega^p(M, E)$ have representations as linear combination of $\alpha_{i_1, \cdots, i_p} \,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \otimes (s_1, \cdots s_p)$.

Definition. The exterior covariant derivative is the map given by extension of

$$\begin{aligned} \mathrm{d}^\nabla \colon \Omega^p(M, E) &\to \Omega^{p+1}(M, E), \\ \alpha \otimes u &\mapsto \mathrm{d}^\nabla(\alpha \otimes u) = \mathrm{d}\alpha \otimes u + (-1)^p \alpha \wedge \nabla u \end{aligned}$$

for $\alpha \in \bigwedge^p \mathrm{T}^* M$, $u \in \Omega^0(E)$.

We want to define an inner product on $\Omega^p(M, E)$. For this, fix a metric g on M and let $(E, \nabla, q) \to M$ be a vector bundle with metric and connection over M.

$$\langle \alpha \otimes u, p \otimes v \rangle = \int_M g(\alpha, p) q(u, v) \,\mathrm{d}v_g$$

is a number. (For this integral to be finite, assume M is compact or work with compactly supported sections.)

Definition. The *exterior covariant codifferential*[1] is the formal L^2-adjoint of d

$$\delta^\nabla \colon \Omega^p(M, E) \to \Omega^p(M, E)$$

such that $\left\langle \mathrm{d}^\nabla(\alpha \otimes u), \beta \otimes v \right\rangle = \left\langle \alpha \otimes u, \delta^\nabla(\beta \otimes v) \right\rangle$.

Remark (Fact). An integration by parts argument shows that δ^∇ exists and, when ∇ is a metric connection, then

$$\delta^\nabla \colon \Omega^1(M, E) \to \Omega^0(M, E), \ \alpha \otimes u \mapsto -\operatorname{tr}_g(\nabla^{\mathrm{T}^* \otimes E} \alpha \otimes u),$$

where for $\Omega^1(M, E) \to \Omega^0(M, \mathrm{T}^* M \otimes \mathrm{T}^* M \otimes E)$, we can take a trace with the metric by choosing an orthonormal basis.

[1]non-standard notation

Definition. A *harmonic p-form* with values in E is an element $\omega_i \in \Omega^p(M,E)$ such that $\delta^\nabla = \delta^\nabla\omega = 0$. As a matter of fact this is equivalent to $\Delta\omega = 0$ for $\Delta := \delta^\nabla \circ \mathrm{d}^\nabla + \mathrm{d}^\nabla \circ \delta^\nabla$ (Consider $\langle \Delta\omega, \omega\rangle$ and utilize the obvious stuff).

1.2. Definition of harmonic maps of 1st variation formula

Let (M,g) and (N,h) be two riemannian manifolds and let $f\colon M \to N$ be a smooth map. Then $\mathrm{d}f\colon \mathrm{T}\,M \to \mathrm{T}\,N$ is an element $\mathrm{d}f \in \Omega^0(\mathrm{Hom}(\mathrm{T}\,M, f^*\,\mathrm{T}\,N)) = \Omega^0(\mathrm{T}^*\,M \otimes f^*\,\mathrm{T}\,N)$.
Next, the metrics g,h induce a metric on $\mathrm{T}^*\,M \otimes f^*\,\mathrm{T}\,N$.

Definition. The energy density of $f\colon M \to N$ is $e(f) := \frac{1}{2}\langle \mathrm{d}f, \mathrm{d}f\rangle_{\mathrm{T}^*\,M\otimes f^*\,\mathrm{T}\,N} = \frac{1}{2}\|\,\mathrm{d}f\|^2$.

Choose co-ordinates $\{x^i\}$ in M and $\{y^i\}$ in N. With respect to these, we have

$$\frac{1}{2}\|\,\mathrm{d}f\|^2 = \frac{1}{2}y^{ij}\partial_i f^*\partial_j f^\beta h_{\alpha\beta}(f).$$

Definition. The *Dirlichlet energy* is given by

$$E\colon C_0^2(M,N) \to \mathbb{R},\ f \mapsto \int_M e(f)\,\mathrm{d}V_g.$$

A *critical map* (or *stationary map*) is a map $f\colon M \to N$ such that for all compactly supported $F\colon M \times (-\varepsilon,\varepsilon) \to N$ C^2-map (variation of f) with $F(x,0) = f(x)$ we have that

$$\delta E(\nu) := \frac{\mathrm{d}}{\mathrm{d}t}E(F)\bigg|_{t=0} = 0 \tag{1.1}$$

for $\nu = \frac{\mathrm{d}}{\mathrm{d}t}F|_{t=0} \in \Omega^0(f^*\,\mathrm{T}\,N)$. The eq. (1.1) is called *first variation in the direction of ν*.

Definition. The map $f\colon (M,g) \to (N,h)$ is called *harmonic*, if it is a critical point for the Dirlichlet energy.

Definition. Let $\mathrm{d}f \in \Omega^1(M, f^*\,\mathrm{T}\,N)$ then $\nabla\,\mathrm{d}f \in \Omega^0(M, \mathrm{T}^*\,M \otimes \mathrm{T}^*\,M \otimes E)$.
The *second fundamental form* of f is $\nabla\,\mathrm{d}f := B_f$, which is a symmetric 2-tensor on M.

Definition. The *tension field* of f is the trace of B_f: $\tau(f) := \mathrm{tr}_g(B_f) \in \Omega^0(M, f^*\,\mathrm{T}\,N)$.

Theorem (1st variation of E). *Let $F\colon M \times (\varepsilon,\varepsilon) \to N$ a variation of f and let $\nu = \frac{\mathrm{d}}{\mathrm{d}t}F|_{t=0}$. Then*

$$\delta E(\nu) = \frac{\mathrm{d}}{\mathrm{d}t}E(F)|_{t=0} = -\int_M \langle \tau(f), \nu\rangle\,\mathrm{d}v_g.$$

Proof. The variation $F\colon M \times (-\varepsilon, \varepsilon) \to N$ yields a pullback connection on $F^* \mathrm{T}\, N$, which shows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} E(F)|_{t=0} &= \frac{1}{2}\int_M \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{d}F, \mathrm{d}F\rangle\,\mathrm{d}V_g|_{t=0} = \int_M \left\langle \nabla_{\frac{\partial}{\partial t}}\mathrm{d}F, \mathrm{d}F\right\rangle \mathrm{d}V_g|_{t=0}\\ &= \int_M \left\langle \nabla^{f^*\mathrm{T}\,N}\nu, \mathrm{d}f\right\rangle \mathrm{d}V_g \overset{(*)}{=} \int_M \left\langle \nu, \delta^{\nabla^{f^*\mathrm{T}\,N}}\mathrm{d}f\right\rangle \mathrm{d}V_g\\ &= -\int_M \langle \nu, \mathrm{tr}_g(\nabla\,\mathrm{d}f)\rangle\,\mathrm{d}V_g = -\int_M \langle \nu, \tau(f)\rangle\,\mathrm{d}V_g,\end{aligned}$$

where $(*)$ follows by a calculation in local co-ordinates. $\square$

Corollary (Fundamental theorem of the calculus of variations). *A C^2-map $f\colon (M,g) \to (N,h)$ is harmonic if and only if $\tau(f) = 0$.*

What does $\tau(f) = 0$ look like?
Fix local co-ordinates $\{x^i\}$ on M and $\{y^j\}$ on N. Then $\mathrm{d}f = \partial_i f^{alpha}\,\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha}$ and thus

$$\begin{aligned}\nabla\,\mathrm{d}f &= \nabla\partial_i f^\alpha\,\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha} = \partial_j\partial_i f^\alpha\,\mathrm{d}x^j \otimes \mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha} + \partial_i f^\alpha \nabla\,\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha}\\ &= A + \partial_i f^\alpha(\nabla\,\mathrm{d}x^i \otimes \frac{\partial}{\partial y^\alpha} + \mathrm{d}x^i \otimes \nabla\frac{\partial}{\partial y^\alpha})\\ &= A + \partial_i f^\alpha(-\Gamma^i_{jk}\,\mathrm{d}x^i \otimes \mathrm{d}x^k \otimes \frac{\partial}{\partial y^\alpha} + \mathrm{d}x^i \partial_j f^\beta \Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial y^\gamma})\\ &= \partial_i\partial_j f^\gamma \Gamma^k_{ij}\partial_k f^\gamma + \Gamma^\gamma_{\alpha\beta}(f)\partial_j f^\alpha \partial_i f^{\beta)}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial y^j}.\end{aligned}$$

Thus $\tau(f) = (\Delta_g f^\gamma + \Gamma^\gamma_{\alpha\beta}(f)\partial_i f^\alpha \partial_j f^\beta g^{ij})$.

2. Example and the Bochner formula (a glimpse of rigidity)

Recall that above we considered C^2-maps $f\colon (M,g) \to (N,h)$ with tension field

$$\tau(f) := \mathrm{tr}_g(\nabla\,\mathrm{d}f) = 0 \in \Omega^0(M, f^*\mathrm{T}\,N).$$

In local co-ordinates $\{x^i\}$ on M and $\{y^\alpha\}$ on N this means [2]

$$\tau(f)^\gamma \frac{\partial}{\partial y^\gamma} = (\Delta_g f^\gamma + \tilde{\Gamma}^\gamma_{\alpha\beta}(f)\partial_i f^\alpha \partial_j f^\beta g^{ij})\partial_\gamma = 0,$$

where $\tilde{\Gamma}$ are the Christoffel symbols on (N,h).

[2] Use roman indices for the M and Greek ones for N.

Example. I. Let $(M, g) = (\mathbb{R}, \mathrm{d}t^2)$ and let $\eta\colon \mathbb{R} \to (N, h)$. From above we know that the Laplace-Beltrami operator here reads

$$\Delta_g f = g^{ij}(\partial_i \partial_j f - \Gamma^k_{ij} \partial_k f) = g^{ij}(\partial_i \partial_j f) = \partial_t^2 f$$

for

$$\Gamma^k_{ij} = \frac{g^{km}}{2}(\partial_i g_{im} + \partial_j g_{im} - \partial_m (g_{ij})$$

($= 0$ if $\{g_{ij}\}$ is constant) and $\{g_{ij}\} = g_{11} = f(\partial_t, \partial_t) = \mathrm{d}t^2(\partial_t, \partial_t) = 1$. Hence

$$\tau(\eta)^\gamma \partial_\gamma = (\ddot{\eta}^\gamma + \tilde{\Gamma}^\gamma_{\alpha\beta}(\eta)\dot{\eta}^\alpha \dot{\eta}^\beta)\partial_\gamma = 0,$$

which is if and only if η is a geodesic, i.e. the covariant derivative along M of the curves speed vanishes: $\frac{\mathrm{D}}{\mathrm{d}t}\dot{\eta} = 0$ and thus $E(\eta)|_a^b = \frac{1}{2}\int_a^b \|\dot{\eta}\|^2 \,\mathrm{d}t$.

II. Now let $f\colon (M, g) \to \mathbb{R}$. Here $\tau(f) = \Delta_g f = 0$.

Proposition. *If M is closed, then the energy harmonic functions are constant.*

Proof. By Green's theorem (integration by parts)

$$\int_M \underbrace{g(\nabla f, \nabla f)}_{=\|\nabla f\|^2} \mathrm{d}V_g = -\int \Delta_g f \cdot f \,\mathrm{d}V_g = 0$$

for $\mathrm{d}V_g = \sqrt{\det(\{g_{ij}\})}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Thus $\|\nabla f\|^2 = 0$ and f must be constant. $\square$

In our example, this shows $\Delta_g f = \lambda f$.

III. Let $f\colon (M, g) \to (N, h)$ be an isometric immersion, i.e. $\mathrm{d}f$ is injective and $g = f^* h = h(\mathrm{d}f, \mathrm{d}f)$. Then we have

$$\begin{aligned} e(f) &= \frac{1}{2}\|\,\mathrm{d}f\|^2 = \frac{1}{2} h_{\alpha\beta} \partial_i f^\alpha \partial_j f^\beta g^{ij} = \frac{1}{2}\partial_i f^\alpha \partial_j f^\beta h(\partial_\alpha, \partial_\beta) g^{ij} \\ &= \frac{1}{2} h(\partial_i f^\alpha \partial_\alpha, \partial_j f^\beta \partial_\beta) g^{ij} = \frac{1}{2} h(\mathrm{d}f(\partial_i), \mathrm{d}f(\partial_j)) g^{ij} = \frac{1}{2} g_{ij} g^{ij} = \frac{m}{2} \end{aligned}$$

and hence $E(f) = \frac{m}{2}\operatorname{Vol}(f)$, where $\operatorname{Vol}(f) = \int_M \mathrm{d}V_{f^*h} = \int_M \mathrm{d}V_g$. This shows that f is critical for E if and only if f is critical for $\operatorname{Vol}\colon \operatorname{Imm}(M, N) \to \mathbb{R}_+$. The latter is clearly if and only if f is a **minimal submanifold**.

Examples of minimal submanifolds in $\mathbb{R}^3$ include the 2-plane, or the helicoid.

2.1. Composition laws for harmonic maps

Consider the composition

$$(M,g) \xrightarrow{f} (N,h) \xrightarrow{u} (Z,b).$$

In general, if f,u are harmonic, this needs not be harmonic again, which can be considered "a bug or a feature".

$$B_{u\circ f}(X,Y) = B_u(\mathrm{d}f(X),\mathrm{d}f(Y)) + \mathrm{d}u(B_f(X,Y))$$

for $X,Y \in \mathrm{T}_p M$ and thus $B_{u\circ f} = \nabla^{\mathrm{T}^* M\otimes(u\circ f)^* \mathrm{T} N}(\mathrm{d}(u\circ f))$. Hence $\tau(u\circ f) = \mathrm{d}(\tau(f)) + \mathrm{tr}_g(f^* B_u)$.
If f is harmonic, then $\tau(u\circ f) = \mathrm{tr}_g(f^* B_u)$.

Proposition. *If $f\colon M\to N$, is harmonic and $u\colon N\to Z$ is totally geodesic, i.e. $B_u = 0$. Then $u\circ f$ is harmonic.*

What if $u\colon N\to\mathbb{R}$ is a function and f is harmonic? Then

$$\tau(u\circ f) = \mathrm{tr}_g(f^* B_u) = \mathrm{tr}_g(f^*(\mathrm{Hess}(u)) = \sum_{i=1}^n f^*(\mathrm{Hess}(u))(E_i,E_i).$$

Recall that a function $u\colon (N,h)\to\mathbb{R}$ is convex, if $\mathrm{Hess}(u)$ is positive definite. If f is harmonic and u is convex, then $\tau(u\circ f) = \nabla_g u\circ f \geq 0$ (these are called *subharmonic functions*).

Theorem. *A map is harmonic if and only if it pulls back germs of convex functions to germs of subharmonic functions.*

There are various useful applications of the "synthetic view" on harmonic functions (e.g. Gromov-Shane).

Theorem. *Suppose (M,g) is closed, connected and (N,h) is 1-connected with non-positive curvature. Then every harmonic map $f\colon (M,g)\to(N,h)$ is constant.*

Proof. The distance function $N\to\mathbb{R}_{\geq 0}, x\mapsto \mathrm{d}_N(p,x)^2$ for every $p\in N$ is actually smooth and strictly convex, e.g. $\mathrm{d}_{\mathbb{R}^n}(0,x)^2 = x_1^2+\cdots+x_n^2$.
In case f is harmonic, we have

$$\Delta_g u\circ f = \tau(u\circ f) = \mathrm{tr}_g(f^* B_u) \geq 0$$

and

$$-\int \|\,\mathrm{d}(u\circ f)\|^2\,\mathrm{d}V_g = \int_M \Delta_g u\circ f\,\mathrm{d}V_g \geq 0.$$

Thus $\|\,\mathrm{d}(u\circ f)\| = 0$ and hence $u\circ f$ is constant. □

2.2. Bochner formulas

Let (E, ∇, a) be a riemannian vector bundle, i.e. a is a metric on E, ∇ is a connection on E preserving a ($\nabla a = 0$) and there is a vector bundle projection map $E \to (M, g)$. Let $\omega \in \Omega^p(M, E)$ and let ∇ be a connection on $\Omega^p(M, E)$.

$$\hat{\nabla}\colon \Omega^p(M, E) \to \Omega^p(M, \mathrm{T}^* M \otimes \mathrm{T}^* M \otimes E), \quad \omega \mapsto ((X, Y) \mapsto \nabla_X \nabla_Y \omega - \nabla_{\nabla_X Y} \omega)$$

The *trace Laplacian* is the operator

$$\nabla^2\colon \Omega^p(M, E) \to \Omega^p(M, E), \quad \omega \mapsto \mathrm{tr}_g(\hat{\nabla}\omega).$$

Recall that the *Hodge Laplacian* was the operator

$$\mathrm{d}^\nabla\colon \Omega^p(M, E) \to \Omega^{p+1}(M, E), \quad \alpha \otimes u \mapsto \mathrm{d}\alpha \otimes u + (-1)^p \alpha \wedge \nabla u.$$

With respect to the L^2-pairing $\beta \otimes v \mapsto \int_M g(\alpha, \beta) a(u, v) \, \mathrm{d}V_g$ it has a formal adjoint

$$\delta^\nabla\colon \Omega^{p+1}(M, E) \to \Omega^p(M, E).$$

The Hodge Laplacian is the degree preserving operator given by $\mathrm{d}^\nabla \circ \delta^\nabla + \delta^\nabla \circ \mathrm{d}^\nabla =: \Delta_a$. The *(generalized) Bochner-Lichnerowicz formula* is given by

$$\nabla_a \omega = -\nabla^2 \omega + S_\omega.$$

for $S_\omega \in \Omega^p(M, E)$ with

$$S_\omega(X_1, \cdots, X_p) = \sum_{k=1}^{p} \sum_{i=1}^{m} (-1)^k (R^{\tilde{\nabla}}(e_i, X_k)\omega)(e_i, X_1, \ldots, \hat{X}_k, \ldots, X_n)$$

for $X_i \in \mathrm{T}_p M$, $m = \dim M$ and $\{e_i\}$ an orthonormal frame around p.[3]

Corollary. *Let $f\colon (M, g) \to (N, h)$ be harmonic. Then*

$$\Delta_g e(f) = \|B_f\|^2 - \sum_{ij} \underbrace{h(R^h(f_* e_i, f_* e_j) f_* e_j, f_* e_i))}_{= \lambda \sec(e_i, e_j)} + \sum_i h(f_*(\mathrm{Ric}^g(e_i)), f_k e_i)$$

for an orthonormal frame $\{e_i\}$.

The key observation for an application of this is that, if Ric^g is a positive operator, then the latter sum is positive.

Theorem (Eells-Sampson). *Let (M, g) be a closed with non-negative Ricci curvature and let (N, h) have non-positive sectional curvature.*

[3] Hat ($\hat{X}_k$), as always, means to omit the k-th term.

(i) Then any harmonic map $f\colon (M,g) \to (N,h)$ is totally geodesic, i.e. $\nabla \mathrm{d} f = B_f = 0$.

(ii) If Ric^g *is positive at any point, then f is constant.*

(iii) If the sectional curvature of (N,h) is strictly negative, then f is constant or $f(M)$ is closed geodesic.

Proof. The first statement easily follows from the corollary and $\int_M \langle \nabla u, \nabla v\rangle \,\mathrm{d}V_g = -\int_M \Delta u \cdot v \,\mathrm{d}V_g$. The second is also not that hard and the last requires some work.□

3. The Eells-Sampson existence theorem

Story: Given two manifolds M, N, is there a best map in a given free homotopy class $\beta \in [M,N]$, where $[M,N]$ denotes the free homotopy classes of smooth maps. From now on, "best" means harmonic with respect to some riemannian metric.

Example. If $M = S^n$, then it is a theorem that every homotopy class $\gamma \in [S^1, N]$ (for N closed) admits a harmonic representative $\gamma\colon S^1 \to (N,h)$, i.e. is a closed geodesic.

Example. What about $\dim(M) \geq 2$. In this case it depends on the curvature of (N,h).
Consider the flat torus $\mathbb{T}^2$ and the round sphere $\mathbb{S}^2$. For a degree 1 map $\mathbb{T}^2 \to \mathbb{S}^2$ there is no harmonic map in the homotopy class (see the book by Lin on geometry of harmonic maps).

Theorem (Eells-Sampson 1964). *Let $(M,g),(N,h)$ be closed manifolds and h with non-positive sectional curvature. Then given any $f\colon M \to N$ C^2-map there exists a harmonic map $u\colon (M,g) \to (N,h)$ such that u is freely homotopic to f.*

Try to take $\tau(u) = 0$ for some $u \sim f$.
In this approach $E\colon C^2(M,N) \to \mathbb{R}, f \mapsto \frac{1}{2}\int_M \|\,\mathrm{d}f\|^2 dV_g$ such that $E(f_n) \to \inf_{f\in C^2} E(f)$, we would have to weaken to topology considering Sobolev spaces $W^{1,2}(M,N)$
The other approach using gradient flow goes as follows. Try to solve initial value problem (IVP). Let $f\colon M \times (0,\infty) \to N$, such that $\frac{\partial f}{\partial t} = \tau(f_t)$ and $f(-,0) = f$.
Recall the first variation of every $f_t\colon M \to N, \frac{\mathrm{d}}{\mathrm{d}t} f_t|_{t=0} = 0$. Then $\delta E(\nu) = \frac{\mathrm{d}}{\mathrm{d}t}E(f_t)|_{t=0} = -\int_M \langle \tau(f), \nu\rangle \, dV_g = -Q(\tau(f),\nu)$, where $\langle -,-\rangle$ is the inner product on $f^*\,\mathrm{T}\,N$ induced by h.
If we manage to solve $\frac{\partial f}{\partial t} = \tau(f)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}E(f_t)|_{t=t_0} = \int_M \langle \tau(f), \tau(f_t)\rangle \,\mathrm{d}V_g \leq 0$$

and equal to zero if and only if $\tau(f_{t_0}) = 0$.
$\frac{\partial f^\gamma}{\partial t} = \Delta_g f^\gamma + \Gamma^\gamma_{\alpha\beta}(f)\partial_i f^\alpha \partial_i f^\beta g^{il}$.

3.1. 1st short time existence

Theorem. *Suppose $f\colon M \to N$ is a C^2-map. Then there exists a $T_{max} > 0$ such that (IVP)*

$$\frac{\partial f_t}{\partial t} = \tau(f_t) \text{ and } f_0 \equiv f$$

has a solution on $[0, T_{max}]$. If $T_{max} < \infty$, then

$$\limsup_{t\nearrow T, x\in M} (f_t) = +\infty.$$

Note that there is no assumption on the curvature.

3.2. Need another Bochner formula

Let (N,h) has non-positive sectional curvature and let M be an m-dimensional manifold. Then we can calculate

$$\begin{aligned}
&\frac{\partial}{\partial t} e(f_t) - \Delta_g e(f_t) \\
&\quad = -\underbrace{\|B_{f_t}\|^2}_{=\nabla \operatorname{d} f_t} - \sum_{i=1}^{n} h\Big(\sum_{j=1}^{m} \operatorname{d} f_t(\mathrm{Ric}^g(e_i,e_j)e_j), \operatorname{d} f_t(e_i)\Big) \\
&\qquad + \underbrace{\sum_{i,j=1}^{m} h(R^h(\operatorname{d} f_t(e_i), \operatorname{d} f_t(e_j)) \operatorname{d} f_t(e_j), \operatorname{d} f_t(e_i)),}_{\leq m}
\end{aligned}$$

where $\mathrm{Ric}^g\colon \mathrm{T}\, M \otimes \mathrm{T}\, M \to \mathbb{R}$ is the Ricci tensor, R^h is the full curvature tensor of (N,h) and $\{e_1, \ldots, e_m\}$ is an orthonormal frame of N.
The latter summand is $\mathrm{const}\,\mathrm{sec}^h(\mathrm{span}(\operatorname{d} f_t(e_i), \operatorname{d} f_t(e_j)))$. We continue[4]

$$\leq -\sum_{i=1}^{m} h\Big(\sum_{j=1}^{m} \operatorname{d} f_t(\mathrm{Ric}^g(e_i,e_j)e_j), \operatorname{d} f_t(e_i)\Big) \leq C \sum_{i,j=1}^{m} h(\operatorname{d} f_t(e_i, \operatorname{d} f_t(e_j)) \leq C e(f_t).$$

Thus we get the following theorem.

Theorem. *If (N,h) has non-positive sectional curvature, then*

$$\frac{\partial}{\partial t} e(f_t) - \Delta_g e(f_t) \leq C e(f_t).$$

[4]TODO: According to Andy, who erased this part very quickly, there should be some mistake somewhere here...

3.3. Moser-Harnack inequality

Let $z_0 = (x_0, t_0) \in M \times (0, T)$ and let $0 < R < \min\{\text{inf radius of } g, t_0\}$. The parabolic cylinder is given as follows

$$P_R(z_0) = \{z = (x, t) \in M \times (0, \infty) \mid \mathrm{d}g(x, x_0) < R, t_0 - R^2 \leq t \leq t_0\}.$$

Theorem (Moser). *Suppose $v \in C^2(P_R(z_0))$ is non-negative and satisfies*

$$\frac{\partial v}{\partial t} - \Delta_g v \leq Cv \text{ for } C > 0.$$

Then there exists a $C_1 > 0$ such that

$$v(z_0) \leq C_1 R^{2-m} \int_{P_R(z_0)} v \, \mathrm{d}V_g \, \mathrm{d}t.$$

If we apply this, we obtain

$$e(f_t)(z_0) \leq CR^{2-n} \iint e(f_t) \, \mathrm{d}V_g \, \mathrm{d}t \leq CR^{2-n} \int_{t_0 - R^2}^{t_0} E(f_t) \, \mathrm{d}t \leq CR^{2-n} E(f) R^2.$$

If $T_{\max} < \infty$, then recall $\limsup_{t \nearrow T, x \in M} e(f_t) = +\infty$, but we have proved $e(f_t)$ is uniformly bounded, hence $T_{\max} = +\infty$.

3.4. Black box # 37

Since $e(f_t)$ is bounded for all time, "elliptic regularity" implies that for all $m > 0$ we have $\|\nabla^m \, \mathrm{d} f_t\| \leq C_m$.

For $f \colon M \times [0, \infty) \to N$ by Arzela-Ascoli, we know that there exists a subsequence $t_k \to \infty$ such that $f(-, t_k) \to u$ for $t_k \to \infty$ (in the sense of C^2-convergence).

We calculate

$$\begin{aligned}\int_0^{t_0} \int_M \left\| \frac{\partial f}{\partial t} \right\|^2 \mathrm{d}V_g \, \mathrm{d}t &= \int_0^{t_0} \int_M \|\tau(f_t)\|^2 \, \mathrm{d}V_g \, \mathrm{d}t = -\int_0^{t_0} \frac{\partial E}{\partial t}(f_t) \, \mathrm{d}t \\ &= -E(f_t) + E(f) \leq E(f) < \infty.\end{aligned}$$

Hence $\limsup_{t_0 \nearrow +\infty} \int_{t_0 - 2}^{t_0} \int_M \|\frac{\partial f}{\partial t}\|^2 \, \mathrm{d}V_g \, \mathrm{d}t = 0$. Now one computes a Bochner formula for $\|\frac{\partial f}{\partial t}\|^2_{C^0}$.

This yields an equality of the following form. For each $0 << 1$ we have and each $t > 0$

$$\|\tau(f_t)\|^2_{C^\alpha(M \times [t-1, t])} = \left\| \frac{\partial f}{\partial t} \right\|^2_{C^\alpha(M \times [t-1, t])} < C(\alpha) \left\| \frac{\partial f}{\partial t} \right\|^2_{L^2(M \times [t-2, t])} \xrightarrow{t \to +\infty} 0.$$

Hence there exists a subsequence t_i such that

$$\|\tau(f_i)\|_{C^\alpha} \xrightarrow{t_i \to \infty} 0$$

and hence $0 = \lim \tau(f_i) = \tau(u)$. Since we have C^2-convergence, we can conclude that $u \sim f$.

4. Harmonic maps and Teichmüller theory

Classically Teichmüller theory belongs to complex analysis and has been studied for a long time from an analytic perspective. But if we have a holomorphic function, then we know its real and imaginary part to be harmonic and we might hope to obtain a more general connection.
Let Σ be a closed, oriented, connected smooth surface of genus ≥ 2. Further let (M, h) be a riemannian manifold and let $f\colon (\Sigma, g) \to (M, h)$ be a smooth map and $U\colon \Sigma \to \mathbb{R}$ a C^∞-function.

Lemma. $\frac{1}{2}\int \|\,\mathrm{d}f\|^2_{g,h}\,\mathrm{d}V_g = E(f, g, h) = E(f, e^{2u}g, h)$.

Proof.

$$\|\,\mathrm{d}f\|^2_{e^{2u}g,h} = e^{-2u}g^{ij}\partial_i f^\alpha \partial_1 f^\beta h_{\alpha\beta}(f) = e^{-2u}\|\,\mathrm{d}f\|^2_{g,h}$$

where $\mathrm{d}V_{e^{2u}g} = \sqrt{\det(e^{2u}g_{ij})}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = e^{2u}\sqrt{\det(g_{ij})}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = e^{2u}\,\mathrm{d}V_g$ and thus

$$\|\,\mathrm{d}f\|^2_{e^{2u}g,h}\,\mathrm{d}V_{e^{2u}g} = \|\,\mathrm{d}f\|^2_{g,h}\,\mathrm{d}V_g. \qquad \square$$

Corollary. *$f\colon (\Sigma, g) \to (M, h)$ is harmonic if and only if $f\colon (\Sigma, e^{2u}g) \to (M, h)$ is harmonic.*

Thus, the notion of harmonicity only depends on the "conformal class" of g.
A *complex structure* on Σ is a maximal atlas of charts into $\mathbb{C}$ with holomorphic (conformal) transition functions. It yields an almost complex structure given by $J\colon \mathrm{T}\,\Sigma \to \mathrm{T}\,\Sigma$ with $J^2 = -\mathrm{id}$.
Let $\mathcal{M}(\Sigma)$ be the space of riemannian metrics on Σ. Then $C^\infty(M)$ acts on (Σ) by $u \cdot g \mapsto e^{2u}g$. Further denote by $\mathcal{C}(\Sigma)$ the set of complex structures on Σ (viewed as $J\colon \mathrm{T}\,\Sigma \to \mathrm{T}\,\Sigma$ as above), which agree with the orientation. In both cases we consider the C^∞-topology on these spaces.

Theorem. *There is a homeomorphism $C^\infty(\Sigma)\backslash\mathcal{M}(\Sigma) \leftrightarrow \mathcal{C}(\Sigma)$.*

Proof (sketch). Consider the map $\mathcal{C}(\Sigma) \to \mathcal{M}(\Sigma)/C^\infty(\Sigma), \sigma \mapsto$ all metrics g such that g defines the same angle as σ.

Another result from geometric analysis yields so-called "isothermal coordinates" $\mathcal{M}(\Sigma)/C^\infty(\Sigma) \to \mathcal{C}(\Sigma)$. For $g \in \mathcal{M}(\Sigma)/C^\infty(\Sigma)$ there exists coordinates $\{\mathrm{d}x, \mathrm{d}y\}$ on about any $p \in \Sigma$ such that $g = e^{2u}(\mathrm{d}x^2 + \mathrm{d}y^2)$. If (x, y) and $(\tilde{x}, \tilde{y})$ are competing isothermal coordinate systems, then the transition function between them is conformal.
Thus we obtain a map $g \mapsto \{$ isothermal coordinate atlas $\}$, where $e^{2u}g$ is mapped to the same atlas as g. □

Denote by $\mathcal{M}_{-1} := \{h \in \mathcal{M} \mid \sec(h) = -1\} \subset \mathcal{M}$ the subspace of hyperbolic metrics.

Theorem (Application of Eells-Sampson / Hartmann). *Given $h \in \mathcal{M}_{-1}$ and a diffeomorphism $f \colon \Sigma \to \Sigma$ there exists (exactly one[5]) a harmonic map $u \colon (\Sigma, \sigma) \to (\Sigma, h)$ such that u is homotopic to f.*

We obtain a map

$$\psi \colon \mathcal{M}_{-1} \to C^\infty(\Sigma, \Sigma), \quad h \mapsto u_h$$

Moreover, for $\mathrm{Diff}_0(\Sigma) = \{\eta \colon \Sigma \to \Sigma \text{ diffeomorphism isotopic to the id}\}$ the map ψ is $\mathrm{Diff}_0(\Sigma)$-equivariant, i.e. $\psi(\eta \cdot h = \eta^* h) = u_h \circ \eta$ for $\eta \in \mathrm{Diff}_0 \Sigma$. Hence we obtain a map $\mathcal{M}_{-1}/\mathrm{Diff}_0 \Sigma \xrightarrow{\psi} C^\infty(\Sigma, \Sigma)/\mathrm{Diff}_0 \Sigma$.
But we can use the theory of Higgs bundles from the right hand side to some useful vector space. For now, we will obtain a way to get around this and directly pass to a nice vector space.
Let $\mathrm{QD}(\sigma) = \{\alpha \mid \alpha \text{ is a holomorphic section of } \mathrm{T}^*_{\mathrm{hol}} \Sigma^{\otimes 2}\}$. In local coordinates z we have $\alpha(z)\,\mathrm{d}z^2$ for α holomorphic.
One consequence of 19th century complex analysis, known as the Riemann-Roch theorem, implies that $\dim \mathrm{QD}(\sigma) = 3g - 3$.

Theorem (Hopf). *If $f \colon (\Sigma, \sigma) \to (M, h)$ is harmonic, then for a metric g in the conformal class of σ we have*

$$f^* h = \alpha\,\mathrm{d}z^2 + e(f) g + \bar{\alpha}\,\mathrm{d}\bar{z}^2,$$

where $\alpha\,\mathrm{d}z^2$ is harmonic and $e(f) = \frac{1}{2}\|\mathrm{d}f\|^2_{g,h}$, i.e. $\alpha = \alpha\,\mathrm{d}z^2 \in \mathrm{QD}(\sigma)$. α is called the Hopf differential.

Theorem (Mike Wolf). *Consider $\Phi \colon \mathcal{M}_{-1}/\mathrm{Diff}_0 \Sigma \to \mathrm{QD}(\sigma), h \mapsto$ Hopf differential of $u_h \colon (\Sigma, \sigma) \to (\Sigma, h)$ is a diffeomorphism.*

The strategy to proof this starts by noticing that $\mathcal{M}_{-1}/\mathrm{Diff}_0 \Sigma$ is a finite dimensional manifold. We want to show that Φ is injective and proper; thus, Φ is a covering map and hence $\mathrm{QD}(\sigma)$ is simply-connected which implies that Φ is a diffeomorphism.

[5] This uniqueness is due to Hartmann

Step 1: Injective (Sampson). $u_h\colon (\Sigma, g) \to (\Sigma, h)$ with $\partial u\colon \mathrm{T}_{\mathrm{hol}}\,\Sigma \to \mathrm{T}\,\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathrm{d}u_{\mathbb{C}}(\frac{\partial}{\partial z})$.
Bochner formula for $\|\partial f\|^2$. Fact 1 (Schoen-Yau). $\|\partial u\|^2$ is nowhere vanishing.

$$\Delta_g \log \|\partial u\|^2 = 2\|\partial u\|^2 - \frac{2}{\|\partial u\|^2}\|\alpha\|_g^2 - 2.$$

Assume: u_1, u_2 with the same Hopf differential

$$\begin{aligned} 0 &\geq \Delta_g(\log \|\partial u_1\|^2 - \log \|\partial u_2\|^2) \\ &= 2\left(\|\partial u_1\|^2 - \|\partial u_2\|^2\right) - 2\|\alpha\|_g^2\left(\frac{1}{\|\partial u_1\|^2} - \frac{1}{\|\partial u_2\|^2}\right) > 0. \end{aligned}$$

Assume: $\|\partial u_1\| > \|\partial u_2\|$.
Thus, at a maximum of $\log \|\partial u_1\|^2 - \log \|\partial u_2\|^2 > 0$. Hence $\|\partial u_1\| \leq \|u_2\|$. Now do the same with $\log \|\partial u_2\| - \log \|\partial u_1\| \implies \|\partial u_1\| = \|\partial u_2\|$. Similar calculations lead to $\|\,\mathrm{d}u_1\|^2 = \|\,\mathrm{d}u_2\|^2$ and another Bochner formula for $\eta \in \mathrm{Diff}_0\,\Sigma$ shows that $u_1 = u_2 \circ \eta$.
Step 2: Properness. Note that the funcationals

$$\begin{aligned} A\colon \mathcal{M}_{-1}/\,\mathrm{Diff}_0\,\Sigma \to \mathbb{R}, &\quad h \mapsto \int \|\alpha\|_g^2\,\mathrm{d}V_g \\ E\colon \mathcal{M}_{-1}/\,\mathrm{Diff}_0\,\Sigma \to \mathbb{R}, &\quad h \mapsto E(u_h) \end{aligned}$$

are connected as follows

$$E(h) - 2\pi(2g-2) \leq A(h) \leq E(h) + 2\pi(2g-2).$$

Thus, $A(h_n) \to \infty$ if and only if $E(h_n) \to \infty$.
Hence Φ is proper if and only if E is proper.

Theorem (Wolf). *E is proper, i.e. it takes a lot of energy to map dissimilar surfaces to one another.*

Then, by the previous discussion E, hence A and hence Φ is proper. Thus, Φ is injective, proper and smooth and therefore a cover, but since $\mathrm{QD}(\sigma)$ is simply connected, it is a diffeomorphism.

Corollary. *$\mathcal{M}_{-1}/\,\mathrm{Diff}_0\,\Sigma$ is a contractible manifold; in fact it is diffeomorphism to C^{3g-3}.*

$\mathcal{C}(\Sigma) \hookrightarrow \mathcal{M}/C^\infty(\Sigma)$. By uniformization, we have a section $\mathcal{M}/C^\infty(\Sigma) \to \mathcal{M}_{-1} \subset \mathcal{M}$ and thus

$$\mathrm{QD}(\sigma) \simeq \mathcal{T}(\Sigma) \simeq \mathcal{C}(\Sigma)/\,\mathrm{Diff}_0 \simeq \mathcal{M}_{-1}/\,\mathrm{Diff}_0(\Sigma) \simeq \mathbb{C}^{3g-3},$$

where $\mathcal{T}(\Sigma)$ is Teichmüller space.

Bibliography

Index

p-forms, 22

based exact, 12
based free, 10
Bochner-Lichnerowicz formula, 27

cellular, 5
complex structure, 31
connection, 21
critical map, 23
CW-complex, 3

degree, 5
Dirlichlet energy, 23

elementary collapse, 16
elementary expansion, 16
equivalent, 19
exterior covariant codifferential, 22

finite, 10
first variation in the direction of ν, 23
free, 10

geometric Whitehead group, 19

harmonic, 23
harmonic p-form, 23
Hodge Laplacian, 27
Hopf differential, 32

Lefschetz number, 5

mapping cone, 10
mapping cylinder, 9
metric connection, 21

n-skeleton, 4

projective, 10
projective module, 8

second fundamental form, 23
simple homotopy equivalence, 16
stationary map, 23
subharmonic functions, 26
suspension, 10

tension field, 23
trace Laplacian, 27

Whitehead group, 9
Whitehead lemma, 9
Whitehead torsion, 12, 16